Lecture 24: Systems of Ordinary Differential Equations

Reading:
Kreyszig Sections: 4.1, 4.2

Systems of Ordinary Differential Equations

The ordinary differential equations that have been treated thus far are relations between a single function and how it changes:

$$F(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x) = 0 \tag{24-1}$$

Many physical models of systems result in differential relations between *several* functions. For example, a first-order system of ordinary differential equations for the functions $(y_1(x), y_2(x), \ldots, y_n(x))$ is:

$$\begin{aligned} \frac{dy_1}{dx} &= f_1(y_1(x), y_2(x), \ldots, y_n(x), x) \\ \frac{dy_2}{dx} &= f_2(y_1(x), y_2(x), \ldots, y_n(x), x) \\ \vdots &= \vdots \\ \frac{dy_n}{dx} &= f_n(y_1(x), y_2(x), \ldots, y_n(x), x) \end{aligned} \tag{24-2}$$

or with a vector notation,

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}, x) \tag{24-3}$$

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Example: The Spread of a MIT Joke

The predator-prey model serves as the classical example of a system of differential equations. This is a (possibly humorous) variant of the predator-prey problem.

Suppose there is a fairly bad joke that circulates around the student population. Students either know the joke or they don't and thus can be divided into two populations:

Jaded, J Knows the joke, and if someone tries to tell it to them, they interrupt with, "Yeah, Yeah. I heard that one. It's pretty, like, stupid."

Naive, N Never heard the joke or has forgotten it.

As the joke spreads, or as students graduate, or students forget the joke, or as new students are admitted to MIT, the populations change.

We will try to construct a model that reflects how the populations change each day.

We will suppose that freshman enter MIT a constant daily rate; in order to keep the population of students regulated, the admissions office accepts freshman at a rate that depends on how many of 4000 slots are open. Therefore, freshman enter MIT, and thus the Naive population at a daily rate of:

$$\frac{dN_{frsh}}{dt} = \frac{4000 - (J + N)}{365} \tag{24-4}$$

Students have a lot of things on their mind (some of which is education) and so they tend to be forgetful. Students who know the joke tend to forget at rate ϕ/year. Suppose that a fraction, ϕ, of the Jaded students forget the joke each year—these students leave the J-group and enter the N-group at a daily rate:

$$\begin{aligned} \frac{dJ_{forg}}{dt} &= \frac{\alpha_A J}{365} = \alpha J \\ \frac{dN_{forg}}{dt} &= -\frac{\alpha_A J}{365} = -\alpha J \end{aligned} \tag{24-5}$$

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It is closely held secret that Susan Hockfield, MIT's president, has an odd sense of humor. At each commencement ceremony, as the proud candidates for graduation approach the president to collect their hard-earned diploma, President Hockfield whispers to the student, "Have you the joke about...?" If the student says, "Yes. I have heard that joke. It is *very* funny!!!" then the diploma is awarded. However, if the student says, "No. But, I am dying to hear it!!!", the president's face drops into a sad frown and the student is asked to leave without collecting the diploma.[16]

Therefore, only students in the J-group can graduate. Let's assume that at any one time, 1/3 of the jaded students have satisfied the graduation requirements, and of this group 99% will graduate:

$$\frac{dJ_{grdt}}{dt} = -\frac{0.99J/3}{365} = -\gamma J \tag{24-6}$$

The joke spreads in proportion to its "funniness coefficient" and the probability that a naive student runs into a jaded student:

$$\begin{aligned} \frac{dJ_{sprd}}{dt} &= -\frac{\phi_A JN}{365^2} = -\phi JN \\ \frac{dN_{sprd}}{dt} &= \frac{\phi_A JN}{365^2} = \phi JN \end{aligned} \tag{24-7}$$

Therefore, an iterative model for the student population that knows the joke is:

$$\begin{aligned} \text{Naive Fraction(Tomorrow)} =& \text{Naive Fraction(Today)} + \text{Change in Naive Fraction} \\ \text{Jaded Fraction(Tomorrow)} =& \text{Jaded Fraction(Today)} + \text{Change in Jaded Fraction} \end{aligned} \tag{24-8}$$

or

$$\begin{aligned} N_{i+1} =& N_i + \frac{4000 - (N_i + J_i)}{365} + \alpha J_i - \phi J_i N_i \\ J_{i+1} =& J_i + \phi J_i N_i - \gamma J_i - \alpha J_i \end{aligned} \tag{24-9}$$

16 This event is an annual source of confusion and embarrassment for the students' proud families—and a source of sadistic amusement to the attending faculty (who have an even stranger sense of humor than Hockfield's).

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