Lecture 24: Systems of Ordinary Differential Equations

Reading:
Kreyszig Sections: 4.1, 4.2

Systems of Ordinary Differential Equations

The ordinary differential equations that have been treated thus far are relations between a single function and how it changes:

$$F(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x) = 0 \tag{24-1}$$

Many physical models of systems result in differential relations between *several* functions. For example, a first-order system of ordinary differential equations for the functions $(y_1(x), y_2(x), \ldots, y_n(x))$ is:

$$\begin{aligned} \frac{dy_1}{dx} &= f_1(y_1(x), y_2(x), \ldots, y_n(x), x) \\ \frac{dy_2}{dx} &= f_2(y_1(x), y_2(x), \ldots, y_n(x), x) \\ \vdots &= \vdots \\ \frac{dy_n}{dx} &= f_n(y_1(x), y_2(x), \ldots, y_n(x), x) \end{aligned} \tag{24-2}$$

or with a vector notation,

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}, x) \tag{24-3}$$

Example: The Spread of a MIT Joke

The predator-prey model serves as the classical example of a system of differential equations. This is a (possibly humorous) variant of the predator-prey problem.

Suppose there is a fairly bad joke that circulates around the student population. Students either know the joke or they don't and thus can be divided into two populations:

Jaded, J Knows the joke, and if someone tries to tell it to them, they interrupt with, "Yeah, Yeah. I heard that one. It's pretty, like, stupid."

Naive, N Never heard the joke or has forgotten it.

As the joke spreads, or as students graduate, or students forget the joke, or as new students are admitted to MIT, the populations change.

We will try to construct a model that reflects how the populations change each day.

We will suppose that freshman enter MIT a constant daily rate; in order to keep the population of students regulated, the admissions office accepts freshman at a rate that depends on how many of 4000 slots are open. Therefore, freshman enter MIT, and thus the Naive population at a daily rate of:

$$\frac{dN_{frsh}}{dt} = \frac{4000 - (J + N)}{365} \tag{24-4}$$

Students have a lot of things on their mind (some of which is education) and so they tend to be forgetful. Students who know the joke tend to forget at rate ϕ/year. Suppose that a fraction, ϕ, of the