E3-1 Estimate how many of the combined pools on the MIT campus of water are contained in a big fluffy cloud on a summer’s day. Report your answer in units of total MIT pool volumes.

To solve this problem, we need a solution in the units of (total water volume contained in a cloud) per (total volume of MIT pools).

According to Scientific American (http://www.scientificamerican.com/article.cfm?id=why-do-clouds-float-when), there is about 1.0 gram of water per cubic meter in a cloud. This value is the water-density of a cloud, water per cubic meter.

According to an article published in the Journal of Climate and Applied Meteorology by B.A. Wielicki and R.M. Welch, “Cumulus Cloud Properties Derived Using Landsat Satellite Data” < http://journals.ametsoc.org/doi/pdf/10.1175/1520-0450(1986)025%3C0261%3ACCPDUL%3E2.0.CO%3B2 >, a typical cumulus cloud is roughly several km tall (take 5 km), and a a few km on each side (let’s take 2 km for a big one). Therefore, the total volume of a typical large cumulus cloud is 2 km × 2 km × 5 km. The mass of water contained in the cloud is
this volume times the density, which is $10^{-3}$ kg/m$^3$.

$$\text{mass}_{\text{h2o}} = (2000)^2 (5000) \times 10^{-3}$$

(* water mass in kg = volume of cloud in m$^3$ times cloud water density, kg/m$^3$ *)

20 000 000

The density of fresh (cloud) water is 1.0 g/cm$^3$, so the volume of water in a cloud is:

$$\text{vol}_{\text{h2o}} = \text{mass}_{\text{h2o}} / 1000 \times 10^{-3}$$

(* water volume, m$^3$ = water mass in kg divided by water density in kg/m$^3$ *)

20 000

This is the (total water volume contained in a cloud). This is the numerator of the desired solution.

There are 3 pools at MIT: the alumni gym pool, the main, olympic-size Z-center pool, and a small Z-center pool. From experience looking at these pools, the Alumni pool is about 40 m $\times$ 15 m $\times$ 2 m deep, the olympic pool is 50 m $\times$ 25 m $\times$ about 3 m deep (it has variable depth, so this includes the very deep diving end of the pool), and the small Z-center pool is roughly 20 m $\times$ 10 m $\times$ 2 m. The total MIT pool volume is:
\*volpools\* = 40 \* 15 \* 2 (\* alumni \*) + \\
50 \* 25 \* 3 (\* Z-center olympic pool \*) + \\
20 \* 10 \* 2 (\* small Z-center \*)

(\* all values in meters, \\
so volpools has units of m\(^3\) \*)

5350

This is (total volume of MIT pools), which is the denominator of the desired result. Combining the numerator and denominator gives

\[ \frac{\text{volh2o}}{\text{volpools}} \]

(\* cloud water volume / MIT pool volume \*)

3.73832

There are about 4 MIT pool-volumes of water contained in a large cumulus cloud.

**I3-1** Solve the following system of two equations for \(x\) and \(y\).

**i. Solve for \(x\) & \(y\)**

Your solution should be handworked; however, the solutions with be provided with *Mathematica*. The function “Solve” can be used to find \(x\) & \(y\) in terms of \(t\) and \(z\).
solution = Solve[{12 == 3 x + 4 y - 5 z + 10 t, 0 == -x + 2 y - 5 z + 20 t}, {x, y}]

\[ \{x \mapsto \frac{12}{5} + 6 \ t - z, \ y \mapsto \frac{6}{5} - 7 \ t + 2 \ z \} \]

A trick to actually define the equation for x as given above by “solution” uses ReplaceAll (/.) and Part ([[ ]]). “xequation” and “yequation” are the expressions for x and y.

\[
\text{xequation} = x \text{/.. solution}[[1]]
\]

\[
\frac{12}{5} + 6 \ t - z
\]

\[
\text{yequation} = y \text{/.. solution}[[1]]
\]

\[
\frac{6}{5} - 7 \ t + 2 \ z
\]

ii. Find the restrictions on t & z

Following the hint from pset #2, use Reduce to find the restrictions on t ans z.
bounds =  
Reduce[xequation > 0 && yequation > 0,  
{t, z}]

\[
t > -\frac{6}{5} \quad \text{and} \quad -\frac{1}{10} (-6 + 35 t) < z < \frac{1}{5} (12 + 30 t)
\]

iii. Plot the region where \( x \& y > 0 \) on the t-z plane

The restriction on \( t \) is simple, it just has to be larger than -6/5, so plot \( z = \frac{1}{10} (-6 + 35 t) \) and \( z = \frac{1}{5} (12 + 30 t) \) for \( t > -6/5 \), and the wedge between these two lines is the desired region, where \( x \& y \) are greater than zero.
RegionPlot[bounds, {t, -2, 10},
{z, -10, 60}, ColorFunction -> Blue,
AxesLabel -> {Style["t", FontSize -> 28],
Style["z", FontSize -> 28]},
PlotLabel ->
Pane[
Style[
"Region where x & y are positive
on the z vs. t plane",
FontSize -> 24], 300]]

Region where x & y are positive on the z vs. t plane
I3-2 For the van der Waals equation of state, introduce non-dimensional variables ($\Pi$, $\Omega$, and $\Theta$) in terms of the critical pressure, volume, and temperature.

i. Units of $\Pi$, $\Omega$, and $\Theta$?

$\Pi$, $\Omega$, and $\Theta$ are dimensionless.

ii. For what values of $\Pi$, $\Omega$, and $\Theta$ does the critical point appear?

When $\Pi$, $\Omega$, and $\Theta$ are equal to 1, the critical point appears.
iii. Plot 5 isotherms on the $\Pi$-$\Omega$ plane.

Recycling our result from pset #2, we found that

$$
\begin{align*}
P_{\text{crit}} &= a / (27 b^2); \\
V_{\text{crit}} &= 3 b; \\
T_{\text{crit}} &= 8 a / (27 b R);
\end{align*}
$$

With the substitutions given in the problem statement, the Van Der Waals gas equation becomes:

$$
 vdwLhs = \text{FullSimplify}[ \\
 ((P + a / V^2) (V - b)) / . \\
 \{T \rightarrow T_{\text{crit}} \theta, V \rightarrow V_{\text{crit}} \Omega, P \rightarrow P_{\text{crit}} \Pi\} \\
] (* \text{ left hand side } *)
$$

$$
\frac{a (-1 + 3 \Omega) (3 + \Pi \Omega^2)}{27 b \Omega^2}
$$

$$
 vdwRhs = \text{FullSimplify}[ \\
 (R T) / . \{T \rightarrow T_{\text{crit}} \theta, V \rightarrow V_{\text{crit}} \Omega, \\
 P \rightarrow P_{\text{crit}} \Pi\} \\
] (* \text{ right hand side } *)
$$

$$
\frac{8 a \theta}{27 b}
$$

set Lhs = Rhs, and solve for $\Pi$: 
\[ \Pi_{vdw} = \]
\[ \text{Simplify[} \quad \Pi \/. (\text{Solve[vdwLhs \[Equal\] vdwRhs, \Pi] // Flatten]) \]
\[ \frac{3 - 9 \Theta + 8 \Theta \Omega^2}{\Omega^2 (-1 + 3 \Omega)} \]

This is the non-dimensionalized van der Waals equation.

Now that we have our non-dimensional expression for \( \Pi \), plot it, here with cold colors grading into hot colors as \( \Theta \) increases:
Show[
  Plot[
    Evaluate[
      Sequence[Table[Pi[v],
        {Theta, {0.8, 0.9, 1, 1.1, 1.2}},
        {Omega, 0.4, 1.2}],
      PlotStyle -> {Purple, Blue, Green, Orange, Red},
      PlotLabel -> Style["Isotherms of Pi",
        FontSize -> 28],
      AxesLabel -> {Style["Omega", FontSize -> 20],
        Style["Pi", FontSize -> 20]}],
    Graphics[
      Red, Text[Style["Theta = 1.2", FontSize -> 16],
        {1.1, 6.5}], Orange,
      Text[Style["Theta = 1.1", FontSize -> 16],
        {1.1, 6.}]], Green,
      Text[Style["Theta = 1", FontSize -> 16],
        {1.1, 5.5}], Blue,
      Text[Style["Theta = 0.9", FontSize -> 16],
        {1.1, 5}], Purple,
      Text[Style["Theta = 0.8", FontSize -> 16],
        {1.1, 4.5}]]]]}
Isotherms of $\Pi$
iv. Plot the surface that represents the van der Waals equation.

```math
Plot3D[\Pi_{vdw}, \{\Theta, 0.2, 1.2\}, \{\Omega, 0.35, 1.1\},
AxesLabel \rightarrow \{Style["\Theta", FontSize \rightarrow 20],
Style["\Omega", FontSize \rightarrow 20],
Style["\Pi", FontSize \rightarrow 20]\},
PlotLabel \rightarrow Style["\Pi(\Theta, \Omega)",
FontSize \rightarrow 28]]
```

v. Find the values of the van der Waals constants a and b for water vapor. Draw the
isotherms near the critical temperature for water vapor.

For water, $a = 5.536 \text{ } L^2 \text{ atm} / \text{ mol}^2 = 0.560935 \text{ } m^4 \text{ N} / \text{ mol}^2$  $b = 0.03049 \text{ } L/\text{ mol} = 0.00003049 \text{ } m^3/\text{ mol}$, from Wikipedia

$$a_{\text{Water}} = 0.560935;$$
$$b_{\text{Water}} = 0.00003049;$$
$$R = 8.31446; \text{ (*) J/mol K (*)}$$
$$t_{\text{crit Water}} = \frac{8 \times a_{\text{Water}}}{27 \times b_{\text{Water}} \times R}$$

655.613

$T_{\text{crit}}$ for water is $\sim 655$ K, so plot isotherms near this value (I chose 550, 600, 650, & 700):
Show[
Plot[
Evaluate[
Sequence[Table[-\(\frac{a_{\text{Water}}}{V^2} + \frac{RT}{-b_{\text{Water}} + V}\),
{T, {550, 600, 650, 700}}],
{V, b_{\text{Water}}, 20 b_{\text{Water}}},
PlotRange -> {{0, 20 b_{\text{Water}}},
10^7 {-0.1, 4}},
PlotStyle -> {Blue, Green, Orange, Red},
PlotLabel ->
Style["Isotherms of P for Water",
FontSize -> 28],
AxesLabel ->
{Style["Volume (L)", FontSize -> 20],
Style["Pressure (Pa)",
FontSize -> 20]}],
Graphics[
{Blue,
Text[Style["T = 550 K", FontSize -> 18],
{0.0002, 1 \times 10^7}], Green,
Text[Style["T = 600 K", FontSize -> 18],
{0.00015, 1.45 \times 10^7}], Orange,
Text[Style["T = 650 K", FontSize -> 18],
{0.00012, 1.85 \times 10^7}], Red,
Text[Style["T = 700 K", FontSize -> 18],
{0.0002, 2.5 \times 10^7}]}]]
vi. What are the units of $\beta$? Think physically about what the sign of $\beta$ means.

$$\beta = -V \frac{dP}{dV}$$

$V$ has units of $m^3/mol$, $dP/dV$ has units of Pressure/molar volume, $N\text{ mol}/m^5$, so $\beta$ must have units $N/m^2 = \text{Pa}$. 
If \( P \) goes down as \( V \) increases (like the real world), then \( \beta \) is positive. If it were not positive, a balloon would expand unstably, or shrink to a point.

**vii. Find a non-dimensional representation of \( \beta \)**

Let \( B = \beta / P_{\text{crit}} \).

\[
\frac{dP}{dV} = \frac{d(P/P_{\text{crit}})}{d(V/V_{\text{crit}})} = \frac{d\Pi}{d\Omega} \frac{V_{\text{crit}}}{P_{\text{crit}}}
\]

\[
B = -V \frac{dP}{dV} / P_{\text{crit}} = -(\Omega V_{\text{crit}}) \frac{d\Pi}{d\Omega} \frac{P_{\text{crit}}}{V_{\text{crit}}} (1 / P_{\text{crit}}) = -\Omega \frac{d\Pi}{d\Omega}
\]

\[
B = -\Omega \left[ \Pi v dw, \Omega \right]
\]

\[
-\Omega \left( \frac{-9 + 16 \Theta \Omega}{\Omega^2 (-1 + 3 \Omega)} - \frac{3 \left( 3 - 9 \Omega + 8 \Theta \Omega^2 \right)}{\Omega^2 (-1 + 3 \Omega)^2} - \frac{2 \left( 3 - 9 \Omega + 8 \Theta \Omega^2 \right)}{\Omega^3 (-1 + 3 \Omega)} \right)
\]

“\( B \)” is the non-dimensionalized equation for \( \beta \).
viii. Plot the values where $\beta$ is negative on the $\Theta$-$\Omega$ plane

```mathematica
Plot3D[B, \{\Omega, 0.2, 1.8\}, \\{\Theta, 0.1, 1.2\},
        PlotRange \to \{All, All, \{0, 60\}\},
        ClippingStyle \to
          \{Lighter@Red, Lighter@Cyan\},
        AxesLabel \to \{Style["\Omega", FontSize \to 20],
                     Style["\Theta", FontSize \to 20],
                     Style["B (non-dimensional $\beta$)",
                            FontSize \to 20]\},
        PlotLabel \to
          Style[
            "The region where $B(\Theta, \Omega)$ is positive",
            FontSize \to 20],
        Epilog \to
          Inset[
            Framed[
              Style[
                "surface & blue = positive | red
                     = negative", 16]],
            \{Right, Bottom\}, \{Right, Bottom\}]
```
Another way to visualize the same thing:
Show[ContourPlot[$\frac{3 - 9 \Omega + 8 \Theta \Omega^2}{\Omega^2 (-1 + 3 \Omega)}$,
{\Omega, 0.3, 1.2}, {\Theta, 0.1, 1.2},
ColorFunction -> "BlueGreenYellow",
AxesLabel -> {Style["\Omega", FontSize -> 20],
Style["\Theta", FontSize -> 20]},
PlotLabel -> Style["B(\Theta, \Omega)",
FontSize -> 20] ],
RegionPlot[B < 0, {\Omega, 0.3, 1.2},
{\Theta, 0.1, 1.2}, PlotStyle -> Pink]]
The blue-green-yellow contours show where B is positive (real world), the pink region is where B is negative (non-physical)

**ix. Repeat the above for water vapor.**

Plot $\beta = -V \frac{dP}{dV}$ for water:

$$\beta = B \text{ } P_{\text{crit}} = B \frac{a_{\text{Water}}}{(27 \text{ } b_{\text{Water}}^2)}$$
\[ \beta_{\text{Water}} = -VD \left[ -\frac{a_{\text{Water}}}{V^2} + \frac{RT}{-b_{\text{Water}} + V}, V \right] \]

\[ -\left( -\frac{8.31446T}{(-0.00003049 + V)^2} + \frac{1.12187}{V^3} \right) V \]

\[ \text{Plot3D}[\beta_{\text{Water}}, \{V, b_{\text{Water}} + 0.0001, 100 b_{\text{Water}}\}, \{T, 10, 1000\}, \text{PlotRange} \to \{\text{All, All, \{0, 10^7\}}\}, \text{ClippingStyle} \to \{\text{Lighter@Red, Lighter@Cyan}\}, \text{AxesLabel} \to \{\text{Style["V (L)"}, \text{FontSize} \to 20], \text{Style["T (K)"}, \text{FontSize} \to 20], \text{Style["\beta_{\text{H}_2\text{O (Pa)}"}, \text{FontSize} \to 20]\}, \text{PlotLabel} \to \text{Style["\beta_{\text{water}}", \text{FontSize} \to 28], \text{Epilog} \to \text{Inset[\ Framed[\ Style[\ "surface & blue = positive | red = negative", 16]\], \{Right, Bottom\}, \{Right, Bottom\}]}} \]
x. What are the units of thermal expansion, $\alpha$?

$$\alpha = \frac{1}{V} \frac{dV}{dT}$$

The units are the same as $1/T$, because the $V$ and $dV$ cancel.
out. Therefore, the units of $\alpha$ are $1/K$.

In the real world, if you heat up a gas at fixed pressure, it expands. Therefore, $\alpha$ is positive. If it were not, then if you heated up the gas in a balloon, it would shrink.

**xi. Find a non-dimensional representation of $\alpha$.**

To non-dimensionalize, multiply by a characteristic temperature to cancel out the units of $\alpha$:

$$A = \alpha \left( \frac{1}{\Omega V_{\text{crit}}} \right) \frac{d\Omega}{d\Theta} \left( \frac{V_{\text{crit}}}{T_{\text{crit}}} \right) T_{\text{crit}} = \left( \frac{1}{\Omega} \right) \frac{d\Omega}{d\Theta}$$

To express $A$, find $\Omega$ as a function of $\Pi$ and $\Theta$ by rearranging the van der Waals equation. It is not a short equation, so the output is suppressed for both $\Omega$ and $A$.

```math
\Omega_{\text{vdw}} = 
Simplify[
    \Omega / . (Solve[vdwLhs == vdwRhs, \Omega] // Flatten)];
```

```math
A = (1 / \Omega_{\text{vdw}}) D[\Omega_{\text{vdw}}, \Theta];
```
xii. Plot where $\alpha$ is negative on the $\Theta$-$\Pi$ plane.

```
Plot3D[A, {\Pi, 0.2, 1.8}, {\Theta, 0.1, 1.2},
       PlotRange -> {All, All, {-1, 0}},
       ClippingStyle -> {Red, Lighter@Cyan},
       AxesLabel -> {Style["\Pi", FontSize -> 20],
                      Style["\Theta", FontSize -> 20],
                      Style["\(A\) (non-dimensional $\alpha$)",
                             FontSize -> 20]},
       PlotLabel -> Style["\(A(\Pi, \Omega)\)",
                           FontSize -> 28],
       Epilog ->
       Inset[
         Framed[
           Style[
             "blue = positive | red & hole = negative",
             16]], {Right, Bottom},
         {Right, Bottom}]]
```
xiii. Repeat for water vapor.

Finding \( V(P,T) \) from the van der Waals equation, taking the derivative with respect to \( T \), and plugging it back into the equation for \( \alpha \), with \( a = a_{\text{Water}} \) and \( b = b_{\text{Water}} \):
\[ V_{vdw} = \]
\[ \frac{V}{.} \]
\[ \text{Solve}\left[(P + \frac{a_{Water}}{V^2})(V - b_{Water}) = RT, \right. \]
\[ V][[3]] \]

Solve::ratnz :

Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. >>
\[
\frac{3.33333 \times 10^{-9} (3049. \cdot P + 8.31446 \times 10^8 T)}{1.19265 \times 10^{-24}} - P
\]
\[
\left(5.77663 \times 10^{34} P - 3.43274 \times 10^{18} (3049. \cdot P + 8.31446 \times 10^8 T)^2 \right) / \left( P \left( 2.85189 \times 10^{29} P^2 + 5.25161 \times 10^{19} P^3 - 3.88848 \times 10^{34} P T + 4.29626 \times 10^{25} P^2 T + 1.17157 \times 10^{31} P T^2 + 1.06493 \times 10^{36} T^3 + 1.29904 \times 10^{11} \right) \right)
\]
\[
\sqrt{\left( 9.6939 \times 10^{44} P^3 + 3.21315 \times 10^{36} P^4 + 2.66258 \times 10^{27} P^5 - 2.19052 \times 10^{42} P^3 T + 2.17822 \times 10^{33} P^4 T - 2.98672 \times 10^{46} P^2 T^2 + 5.93988 \times 10^{38} P^3 T^2 + 5.39925 \times 10^{43} P^2 T^3 \right)}^{1/3} + \]
\[
\frac{1}{P} 2.71397 \times 10^{-12} \left( 2.85189 \times 10^{29} P^2 + 5.25161 \times 10^{19} P^3 - 3.88848 \times 10^{34} P T + 4.29626 \times 10^{25} P^2 T + 1.17157 \times 10^{31} P T^2 + 1.06493 \times 10^{36} T^3 + 1.29904 \times 10^{11} \right)
\]
\[
\sqrt{\left( 9.6939 \times 10^{44} P^3 + 3.21315 \times 10^{36} P^4 + 2.66258 \times 10^{27} P^5 - 2.19052 \times 10^{42} P^3 T + 2.17822 \times 10^{33} P^4 T - 2.98672 \times 10^{46} P^2 T^2 + 5.93988 \times 10^{38} P^3 T^2 + 5.39925 \times 10^{43} P^2 T^3 \right)}^{1/3}
\]

\[
\alpha_{\text{Water}} = \left( \frac{1}{V_{\text{vdw}}} \right) D[V_{\text{vdw}}, T];
\]
Plot3D[\(\alpha_{\text{water}}, \{P, 0, 10^8\}, \{T, 10, 1000\}\),
PlotRange \rightarrow \{\text{All, All, \{0, 0.04\}}\},
ClippingStyle \rightarrow
\{\text{Lighter@Red, Lighter@Cyan}\},
MaxRecursion \rightarrow 5,
AxesLabel \rightarrow
\{\text{Style["P (Pa)", FontSize \rightarrow 20]},
\text{Style["T (K)", FontSize \rightarrow 20]},
\text{Style["\alpha_{H_2O} (1/K)", FontSize \rightarrow 20]}\},
PlotLabel \rightarrow \text{Style["\alpha_{\text{water}}", FontSize \rightarrow 32]},
Epilog \rightarrow
\text{Inset[}
\text{Framed[}
\text{Style[}
\text{"blue & surface = positive | red}
\text{ & hole = negative", 16]},
\{\text{Right, Bottom}, \{\text{Right, Bottom}}\]}\]
The plot is not well resolved near the non-physical region, where $\alpha$ is negative. The peaks and valleys are an artifact of how Mathematica generates plots, not a feature of $\alpha$.

**I3-3 This problem is an example of how to interact with Mathematica and infer the nature**
of general solutions. Consider the systems of linear springs with differing spring constants \((k_i)\) and “zero-strain” lengths \((L_i)\).

i. For the system of two springs, construct a matrix that multiplies the unknowns \((x_1, x_2)\) and gives the right-hand-side vector at equilibrium. Solve this matrix equation to obtain the equilibrium positions \(x_1\) and \(x_2\) in terms of the applied force.

For 2 springs:

\(F_2\) is set to be some known applied force. What are the equilibrium positions of the middle node, \(x_1\), and the end node \(x_2\), and what is the force on the first spring, \(F_1\)?

There are 3 unknowns. Therefore, 3 equations are required to solve. These equations are:

1. Hooke’s law for spring 1: \(F_1 = -k_1 \, dx_1\)
2. Hooke’s law for spring 2: \(F_2 = -k_2 \, dx_2\)
3. Force balance on the center node (this node is stationary at equilibrium, so the net force is zero): \(F_1 = F_2 \implies k_1 \, dx_1 = k_2 \, dx_2\)

\[dx_1 = x_1 - L_1 - x_0\]
\[dx_2 = x_2 - L_2 - x_1\]
In all, the 3 equations, with some algebraic rearrangement, are:

1:  \( F_1 + k_1 x_1 = k_1 L_1 + k_1 x_0 \)
2:  \(-k_2 x_2 + k_2 x_1 = F_2 - k_2 L_2 \)
3:  \((k_1 + k_2) x_1 - k_2 x_2 = k_1 L_1 + k_1 x_0 - k_2 L_2 \)

Writing them in a more suggestive form:

1:  \[ F_1 + k_1 x_1 + 0 x_2 = k_1 L_1 + k_1 x_0 \]
2:  \[ 0 F_1 + k_2 x_1 - k_2 x_2 = F_2 - k_2 L_2 \]
3:  \[ 0 F_1 + (k_1+k_2) x_1 + -k_2 x_2 = k_1 L_1 + k_1 x_0 - k_2 L_2 \]

This now looks like the result of a matrix multiplication:

\[
\begin{pmatrix}
1 & k_1 & 0 \\
0 & k_2 & -k_2 \\
0 & k_1 + k_2 & -k_2
\end{pmatrix}
\begin{pmatrix}
F_1 \\
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
k_1 L_1 + k_1 x_0 \\
F_2 - k_2 L_2 \\
k_1 L_1 + k_1 x_0 - k_2 L_2
\end{pmatrix}
\]

- **Solving for** \( x_1, x_2, \) and \( F_1 \) **by solving the system of equations**

```math
Clear[k, x, F, eq]
```

The first equation:  \( F_1 = k_1 dx_1 \)

\[
\]

\[
\]

The second equation:  \( F_2 = k_2 dx_2 \)


The third equation: \( F_1 = F_2 \):


Solve these 3 equations together:

\[
\text{systemResult} = \\
\text{Solve}[[\text{eq}[1], \text{eq}[2], \text{eq}[3]], \\
\{x[1], x[2], F[1]\}][[1]]
\]

\[
\begin{align*}
F[1] & \rightarrow F[2]\}
\end{align*}
\]
\[
\begin{align*}
x1\text{sys} &= x[1] \/. \text{systemResult} \\
x2\text{sys} &= x[2] \/. \text{systemResult} \\
F1\text{sys} &= F[1] \/. \text{systemResult} \\
\end{align*}
\]

\[
F[2]
\]

This gives the positions of the nodes and F1 in terms of F2, x0, L1, L2, k1, and k2. Note that F1 = F2 falls out.

- **Solving for x1, x2, and F1 using direct input of matrix and b-vector**

\[
\text{matrixA} = \{(1, k[1], 0), (0, k[2], -k[2]), (0, k[1] + k[2], -k[2])\}; \\
\text{MatrixForm}[\text{matrixA}]
\]

\[
\begin{pmatrix}
1 & k[1] & 0 \\
\end{pmatrix}
\]
unknownVector = {F[1], x[1], x[2]};
MatrixForm[unknownVector]

\[
\begin{pmatrix}
F[1] \\
x[1] \\
x[2]
\end{pmatrix}
\]

vectorB = {k[1] L[1] + k[1] x[0],
MatrixForm[vectorB]

\[
\begin{pmatrix}
\end{pmatrix}
\]

Solve the linear system:

\[
\text{linearResult} = \text{LinearSolve}[\text{matrixA}, \text{vectorB}]
\]

\[
\begin{align*}
F[2], & -\frac{F[2]}{k[1]} + L[1] + x[0], \quad \frac{1}{k[1] k[2]} \\
\end{align*}
\]

Define the outputs:
\[
\begin{align*}
x_{1\text{lin}} &= \text{linearResult}[[2]] \\
x_{2\text{lin}} &= \text{linearResult}[[3]] \\
F_{1\text{lin}} &= \text{linearResult}[[1]] \\
\end{align*}
\]
\[
-\frac{F[2]}{k[1]} + L[1] + x[0]
\]
\[
\]
\[
F[2]
\]
Are the results equivalent?

\[
\text{Simplify}[x_{1\text{sys}} == x_{1\text{lin}}] \\
\text{Simplify}[x_{2\text{sys}} == x_{2\text{lin}}] \\
\text{Simplify}[F_{1\text{sys}} == F_{1\text{lin}}]
\]

\[
\text{True} \\
\text{True} \\
\text{True}
\]

This shows that solving the matrix form is equivalent to solving the system of equations.

- **Solving for** \(x_1\), \(x_2\), and \(F_1\) **using** *Mathematica* **to generate the A matrix and b vector**
The same three equations are rearranged so that the left-hand side (LHS) is zero

\[
\begin{align*}
\end{align*}
\]

The list of unknowns:

\[
\text{vars} = \{ x[1], x[2], F[1] \}
\]

\[
\{ x[1], x[2], F[1] \}
\]

This function, row[i], generates the coefficients of each variable for a given equation. These are the coefficients with construct the matrix A.

\[
\text{row}[i_] := \\
\text{Table}[\text{Coefficient}[\text{zeroEq}[i], \text{vars}[[j]]], \\
\{j, 1, 3\}]
\]
Compute the vector of constants, \( \mathbf{b} \), or RHS of the linear algebra equation, by finding the terms that do not depend on the vars using “Thread” (note the minus sign, because these terms are shifted to the opposite side of the equals sign)

\[
\mathbf{b} = \begin{align*}
\{-k[1], 0, -1\} \\
\{k[1] + k[2], -k[2], 0\} \\
\{k[2], -k[2], 0\}
\end{align*}
\]

Compute the matrix, \( \mathbf{A} \), using the function defined earlier, “row”

\[
\mathbf{A} = \begin{align*}
\mathbf{row}[1] \\
\mathbf{row}[2] \\
\mathbf{row}[3]
\end{align*}
\]

Finally, invert \( \mathbf{A} \) and matrix-multiply it into \( \mathbf{b} \) to arrive at the solution
\[
\text{altResult} = \text{Simplify}[\text{Inverse}[\text{matrix}]\cdot \text{rhs}]
\]
\[
\left\{-\frac{F[2]}{k[1]} + L[1] + x[0],
\right.
\]
\[
\]

\[
x1alt = \text{altResult}[[1]];
x2alt = \text{altResult}[[2]];
F1alt = \text{altResult}[[3]];
\]

Demonstration that this alternative solution is identical to the previous two:

\[
\text{Simplify}[x1sys == x1alt]
\]
\[
\text{Simplify}[x2sys == x2alt]
\]
\[
\text{Simplify}[F1sys == F1alt]
\]

True

True

True

ii. i. For the system of three springs, construct a matrix that multiplies the unknowns \((x_1, x_2, x_3)\) and gives the right-
hand-side vector at equilibrium. Solve this matrix equation to obtain the equilibrium positions in terms of the applied force.

For 3 springs:
exactly the same drill, except there are 5 unknowns: F1, F2, x1, x2, x3; and there are 5 equations: Hooke’s law for each spring, and force balance on the middle nodes x1 and x2. Using Mathematica to compute the matrix A and vector b, the solution is easily obtained.

The same three equations are rearranged so that the left-hand side (LHS) is zero
(* hooke's law *)


(* force balances *)


\( k[1] \left( -L[1] - x[0] + x[1] \right) - \\


The list of unknowns:

\[ \text{vars2} = \{ x[1], x[2], x[3], F[1], F[2] \} \]

\[ \{ x[1], x[2], x[3], F[1], F[2] \} \]

This function, row2[i], generates the coefficients of each variable for a given equation. These are the coefficients with construct the matrix A.
row2[i_] := 
Table[Coefficient[zeroEq2[i], vars2[[j]]], 
{j, 1, Length[vars2]}]

row2[1]

{-k[1], 0, 0, -1, 0}

Compute the vector of constants, b, or RHS of the linear algebra equation, by finding the terms that do not depend on the vars using “Thread” (note the minus sign, because these terms are shifted to the opposite side of the equals sign)

rhs2 = 
Table[-zeroEq2[i] /. Thread[Rule[vars2, 0]], 
{i, 1, Length[vars2]}]

{k[1] (-L[1] - x[0]),

Compute the matrix, A, using the function defined earlier, “row”
\textbf{matrix2} = \\
\text{Table[} \\
\quad \text{Table[Coefficient[zeroEq2[i], vars2[[j]]],} \\
\quad \{j, 1, \text{Length[vars2]}\}], \\
\quad \{i, 1, \text{Length[vars2]}\}] \\
\{\{-k[1], 0, 0, -1, 0\}, \{k[2], -k[2], 0, 0, -1\}, \\
\quad \{0, k[3], -k[3], 0, 0\}, \\
\quad \{k[1] + k[2], -k[2], 0, 0, 0\}, \\
\quad \{-k[2], k[2] + k[3], -k[3], 0, 0\}\}

Finally, invert A and matrix-multiply it into b to arrive at the solution

\textbf{result2} = \text{Simplify[Inverse[matrix2].rhs2]} \\
\left\{-\frac{F[3]}{k[1]} + L[1] + x[0], \\
F[3] \left(-\frac{1}{k[1]} - \frac{1}{k[2]} - \frac{1}{k[3]}\right) + L[1] + \\
L[2] + L[3] + x[0], F[3], F[3]\right\}

\textbf{x1part2} = result2[[1]]; \\
\textbf{x2part2} = result2[[2]]; \\
\textbf{x3part2} = result2[[3]]; \\
\textbf{F1part2} = result2[[4]]; \\
\textbf{F2part2} = result2[[5]];

An interesting experiment: investigate whether the solution with
2 springs is the same as the solution with 3 springs:

\[
\text{Simplify}[x_1\text{part2} = x_1\text{lin}]
\]
\[
\text{Simplify}[x_2\text{part2} = x_2\text{lin}]
\]
\[
\text{Simplify}[F_1\text{part2} = F_1\text{lin}]
\]

\[
\frac{F[2] - F[3]}{k[1]} = 0
\]

\[
\]

\[
\]

We see from the solution to part 2, \(F_1 = F_2 = F_3\). Assume that the applied force is the same in parts 1 and 2, so \(F_2 = F_3\). Comparing the solutions for \(x_1\) from part 1 and part 2 of this problem yields “\(F_2 - F_3 = 0\)” But, \(F_2 = F_3\), so the \(x_1\) position is identical whether there are 2 or 3 springs. For \(x_2\), we again see \(F_2 - F_3\), which is zero, so the position is the same in parts 1 and 2. Finally, we see the forces are all the same as well. In short, if the applied force is the same, each spring acts independently - it does not matter how many springs are in the chain.

iii. By inspecting the solutions above, write a general function which takes a list of \(k_i\) and \(L_i\) and returns the positions \(x_i\).

By looking at part 2, we see that Hooke’s law always takes the form:

\[
0 = k_i (x_i - x(i-1) - L_i) - F_i
\]
and if there are n springs, then we have n equations.

The force balances always take the form:
\[ 0 = k_i(x_i - x_{i-1}) - L_i - k_{i+1}(x_{i+1} - x_i - L_{i+1}) \]
and if we have n springs, we generate n-1 equations.

So, we have in total 2n-1 equations, and 2n-1 variables: x1, x2, ...
... xn, F1, F2, ... F(n-1).

The function “springSolver” simply does everything that was shown in part 2, and puts each step inside a Module (so that variable names are local). It takes a list of k’s and L’s, and outputs x’s and F’s.

```math
springSolver[kis_, lis_] := 
Module[{n, listOfHookesEqns, 
      listOfBalanceEqns, eqns, vars, 
      matrixA, vectorb, solution}, 

(* determine how make springs there are *)

n = Length[kis];

(* find the equations in the form 0 = LHS, and generate a list of LHS's *)
listOfHookesEqns = Table[
  -kis[[i]] (x[i] - x[i-1] - lis[[i]]) - 
  F[i], {i, n}];

listOfBalanceEqns = Table[
  kis[[i]] (x[i] - x[i-1] - lis[[i]]) - 
  kis[[i+1]] 
  (x[i+1] - x[i] - lis[[i+1]]),
  {i, n-1}];

(* join the two lists to get all 2n- 
```
1 equations together *)
eqns = Join[listOfHookesEqns, listOfBalanceEqns];

(* write the list of variables *)
vars = Join[Table[x[i], {i, n}], Table[F[i], {i, n - 1}]];

(* generate the matrix *)
matrixA =
  Table[
    Table[Coefficient[eqns[[i]], vars[[j]]], {j, Length[vars]}], {i, Length[eqns]}];

(* generate the RHS, b *)
vectorb =
  Table[-eqns[[i]] /. Thread[Rule[vars, 0]], {i, Length[vars]}];

(* compute the solution *)
solution = LinearSolve[matrixA, vectorb];

(* output the list of unknowns in in the following format: {{x[1], value}, {x[2], value}, ... {F[n-1], value}} *)
Transpose[{vars, solution}] ]

Test springSolver by comparing its solution for 2 springs to the solution to part 1:
\[ \text{soln} = \text{springSolver}\{\{k[1], k[2]\}, \\
\{L[1], L[2]\}\} \]
\[ \text{x1sol} = \text{soln}[[1, 2]]; \]
\[ \text{x2sol} = \text{soln}[[2, 2]]; \]
\[ \text{F1sol} = \text{soln}[[3, 2]]; \]

\[ \{\{x[1], -\frac{F[2]}{k[1]} + L[1] + x[0]\}, \{x[2], -\frac{F[2]}{k[1]} + \\
\[ \{F[1], F[2]\}\} \]

\[ \text{Simplify}[\text{x1sol} \Rightarrow \text{x1lin}] \]
\[ \text{Simplify}[\text{x2sol} \Rightarrow \text{x2lin}] \]
\[ \text{Simplify}[\text{F1sol} \Rightarrow \text{F1lin}] \]

True

True

True

The comparisons all yielded true, so the solutions obtained in part 1 are identical to those produced by the function springSolver.