Problem Set 5: Due: 29 Oct. (10AM)

Individual assignments should be a combination of your hand-worked solutions and other printed material—they should be placed in the mailbox outside Prof. Carter’s door. Email group assignments to 3016-psets(the symbol at)pruffle.mit.edu

For the individual problems indicated as “Handworked”, you should work your solutions by hand and show your work. Print the results of software-worked solutions, and staple them to your hand-worked assignments before turning them in.

There are no group problems for this problem set.

Estimation Exercise E5-1

(State your assumptions, identify any source material, and show your reasoning in such a way that it can be easily followed.)

25 points

MIT undergraduates have access to copious free food. Estimate how the total number of calories that are consumed by an average undergraduate in four academic years. Report your estimate in megacalories (i.e., 1 megacalorie is 1000 “food calorie”).
A fluid inside a spherical container: the goal is to calculate the area of the fluid/container interface. The triple line is the curve which defines where the fluid, vapor, and container are all in contact.

The area of the interface between the container and the fluid can be written as the integral:

\[
A_{f/c} = \int \int_{\text{container}} dA \\
= \int \int_{\text{container}} \hat{n} \cdot \vec{dA} \\
= \int \int_{\text{container}} \frac{x \hat{i} + y \hat{j} + z \hat{k}}{R} \cdot \vec{dA}
\]

where \( R \) is the radius of the sphere. (Explain the last step that relates \( \hat{n} \) to the fraction.)

Show that the surface area can be calculated by a lower dimensional integral over the triple line:

\[
A_{f/c} = \int_{\text{triple line}} \vec{w} \cdot \vec{d}s + \text{constant}
\]

where

\[
\vec{w} = \frac{R^2 z}{(x^2 + y^2)(x^2 + y^2 + z^2)^{1/2}} (y \hat{j} - x \hat{i})
\]

and the constant is a “constant of integration” which can be determined by integrating for a triple line that lies along the equator—for this you know that the surface area must be \( 2\pi R^2 \).

Note: the direction of the integration \( ds \) matters in Stokes’s theorem. The direction of \( ds \) determines whether the integral is over the “covered” or the “uncovered” part of the sphere.
Individual Exercise I5-2
50 points

i: Compute the area of a spherical cap where the cap lies above the plane \( z = R \sin(\alpha) \).

ii: Visualize the curve \((x, y, z) = R(\cos[\frac{\pi}{2} \cos^2(nt)] \cos(t), \cos[\frac{\pi}{2} \cos^2(nt)] \sin(t), \sin[\frac{\pi}{2} \cos^2(nt)])\) (where \( n \) is an integer between 1 and 10 inclusive) and a sphere of radius \( R \) centered on the origin.

iii: Compute the surface area of the portion of the sphere that lies below the curve defined immediately above.

Individual Exercise I5-3
100 points

The Cartesian displacement field due to an edge dislocation located at \( x = y = 0 \) is given by:

\[
\begin{align*}
    u(x, y) &= \frac{b}{2\pi} \left( \tan^{-1} \frac{y}{x} + \frac{1}{2(1 - \nu)} \frac{xy}{r^2} \right), \\
    v(x, y) &= \frac{b}{2\pi} \left( \frac{1 - 2\nu}{2(1 - \nu)} \log \frac{b}{r} + \frac{1}{2(1 - \nu)} \frac{y^2}{r^2} \right)
\end{align*}
\]

where \( b \) is the Burger’s vector (usually on the order of a lattice vector), \( r = \sqrt{x^2 + y^2} \), and \( \nu \) is the material’s Poisson ratio which is usually between 0 and 1/2.

The vector field \((u(x, y), v(x, y))\) are the displacements of a point relative to a dislocation-free lattice. In other words, an atom at a position \((x, y)\) in a dislocation-free lattice would be located at \((x + u(x, y), y + v(x, y))\) in a lattice with a dislocation.

Strain tensor fields can be calculated from vector displacement fields. Strains are illustrated in Lecture 10. For this two-dimensional displacement field, the strain tensor \( \epsilon \) is given by:

\[
\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} \end{pmatrix}
\]

Note that \( \epsilon \) is symmetric \((\epsilon_{xy} = \epsilon_{yx})\)—these off-diagonal terms are called the shear strains and they indicate how much the material is twisting.

i: Visualize the positions of atoms in a square lattice due to an edge dislocation located at \( x = y = 0 \). As a starting point, you can use the following code to construct a pristine lattice:

```plaintext
lattice = Flatten[ Table[{0.5, 0.5} + {i, j}, {i, -6, 6}, {j, -6, 6}], 1];
Graphics[GraphicsComplex[lattice, Table[Disk[i, 1/4], {i, 1, Length[lattice]}]]]
```

ii: Compute the strain field and visualize it.

iii: The two-dimensional “dilation” (i.e., the relative increase in local area) is given by the trace of the strain tensor. Visualize this two-dimensional dilation for the edge dislocation.