Lecture 12: Multivariable Calculus

Reading:
Kreyszig Sections: 9.5, 9.6, 9.7 (pages 389–398, 400–403, 403–409)

The Calculus of Curves

In the last lecture, the derivatives of a vector that varied continuously with a parameter, \( \vec{r}(t) \), were considered. It is natural to think of \( \vec{r}(t) \) as a curve in whatever space the vector \( \vec{r} \) is defined. The most familiar example is a curve in the plane: the two values \((x(t), y(t))\) are mapped onto the plane through values as \( t \) sweeps through its range \( t_{\text{initial}} \leq t \leq t_{\text{final}} \). A curve in three-dimensional Cartesian space is the mapping of three values \((x(t), y(t), z(t))\); and in cylindrical coordinates it is: \((r(t), \theta(t), z(t))\). In general, a curve is represented by \( N \) coordinates, as a single parameter (i.e., \( t \)) takes on a range of numbers—the \( N \) coordinates form the embedding space.

Objects that have more dimensions than curves need more parameters. The number of parameters is the dimensionality of the object and the number of coordinates is the dimensionality of the embedding space. What we naturally call a surface is a two dimensional object embedded in a three-dimensional space—for example, in Cartesian coordinates \((x(u, v), y(u, v), z(u, v))\) is a surface.

The two-dimensional surface \((x(u, v), y(u, v), z(u, v))\) can itself become an embedding space for lower dimensional objects; for example, the curve \((u(t), v(t))\) is embedded in the surface \((u, v)\) which itself is embedded in \((x, y, z)\). In other words, the curve \((x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))\) can be considered to be embedded in \((u, v)\), or embedded in \((x, y, z)\) and constrained to the surface \((x(u, v), y(u, v), z(u, v))\).

In higher dimensions, there are many more possibilities and we can make a few introductory remarks about the language that is used to describe them. For application to physical problems, these considerations indicate the number of degrees-of-freedom that are available and the conditions that a system is over-constrained. An \( N \)-dimensional surface (sometimes called a hyper-surface) embedded in an \( M \)-dimensional space is said to have codimension \( M - N \). Some objects cannot be embedded in a higher dimensional space; these are called non-embeddable, and examples include the Klein bottle which cannot be embedded in our three-dimensional space.
An example is constructed that visualizes a two-dimensional surface in three dimensions and then visualizes a one-dimensional curve constrained to that surface.

Create an example function that returns a position \( \{x, y, z\} \) as a function of two parameters:

\[
\text{FlowerPot}(u, v) := \begin{cases} 
(3 + \cos(v)) \cos(u), \\
(3 + \cos(v)) \sin(u), \\
3 \sin(u) + (3 + \cos(v)) \sin(v)
\end{cases}
\]

1: \textit{FlowerPot} takes two arguments and returns a vector. As the arguments sweep through domains, the vector will trace out a surface.

2: Using the \texttt{ParametricPlot3D}, the surface is visualized. Here we use \texttt{Directive} to make the surface brown with \texttt{Opacity} to make the surface about 40% transparent. \texttt{Specularity} is used to make the surface text look a bit shiny here. The resulting graphics are assigned to the symbol \texttt{Flowerplot}.

3: \textit{Vines} takes a single argument and then calls \textit{FlowerPot} with two arguments that are functions of that single argument—the result must be a curve embedded in the surface. In this case, the function is repeated and is scaled in and out a little, so the curves will be visible later.

4: Here, both the embedded curve and the surface are shown together.

Because the derivative of a curve with respect to its parameter is a tangent vector, the unit tangent can be defined immediately:

\[
\hat{u} = \frac{d\vec{r}}{dt} \left/ \left\| \frac{d\vec{r}}{dt} \right\| \right.
\]

(12-1)

It is convenient to find a new parameter, \( s(t) \), that would make the denominator in Eq. 12-1 equal
to one. This parameter, \( s(t) \), is the arc-length:

\[
s(t) = \int_{t_0}^{t} ds = \int_{t_0}^{t} \sqrt{dx^2 + dy^2 + dz^2} = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt
\]

and with \( s \) instead of \( t \),

\[
\hat{u}(s) = \frac{d\vec{r}}{ds} \quad (12-3)
\]

This is natural because \( \|\vec{r}\| \) and \( s \) have the same units (i.e., meters and meters, feet and feet, etc) instead of, for instance, time, \( t \), that makes \( d\vec{r}/dt \) a velocity and involves two different kinds of units (e.g., furlongs and hours).

With the arc-length \( s \), the magnitude of the curvature is particularly simple,

\[
\kappa(s) = \left\| \frac{d\hat{u}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\| \quad (12-4)
\]

as is its interpretation—the curvature is a measure of how rapidly the unit tangent is changing direction.

Furthermore, the rate at which the unit tangent changes direction is a vector that must be normal to the tangent (because \( d(\hat{u} \cdot \hat{u}) = 0 \)) and therefore the unit normal is defined by:

\[
\hat{p}(s) = \frac{1}{\kappa(s)} \frac{d\hat{u}}{ds} \quad (12-5)
\]

There are two unit vectors that are locally normal to the unit tangent vector \( \hat{u}'(s) \) and the curve unit normal \( \hat{p}(s) \times \hat{u} \) and \( \hat{u}(s) \times \hat{p} \). This last choice is called the unit binormal, \( \hat{b} \equiv \hat{u}(s) \times \hat{p} \) and the three vectors together form a nice little moving orthogonal axis pinned to the curve. Furthermore, there are convenient relations between the vectors and differential geometric quantities called the Frenet equations.

**Using Arc-Length as a Curve’s Parameter**

However, it should be pointed out that—although re-parameterizing a curve in terms of its arc-length makes for simple analysis of a curve—achieving this re-parameterization is not necessarily simple.

Consider the steps required to represent a curve \( \vec{r}(t) \) in terms of its arc-length:

**integration** The integral in Eq. 12-2 may or may not have a closed form for \( s(t) \).

If it does, then we can perform the following operation:

**inversion** \( s(t) \) is typically a complicated function that is not easy to invert, i.e., solve for \( t \) in terms of \( s \) to get a \( t(s) \) that can be substituted into \( \vec{r}(t(s)) = \vec{r}(s) \).

These difficulties usually result in treating the problem symbolically and resorting to numerical methods of achieving the integration and inversion steps.
Lecture 12 MATHEMATICA® Example 2

Calculating arclength


Examples of computing a curve’s arc-length $s$ from the relation $ds = |d\vec{x}| = \sqrt{dx^2 + dy^2 + dz^2}$ are presented.

1–2: Two example functions of a single parameter $t$ that return a vector $\vec{x}$ defined for this example.

3: Here, the tangent-vector for the function, $\text{PrettyFlower}$ defined above, is computed.

4: This is an attempt to find a closed-form solution for arc-length $s(\tau)$:

$$s(\tau) = \int_0^\tau \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \, dt.$$ A closed-form solution doesn’t exist.

5: However, a closed-form solution does exist for the arc-length of the $\text{Bendy}$ function defined earlier. If the closed-form $s(t)$ could be inverted (i.e., $t(s)$) then the curve $c(t)$ could be expressed in terms of its natural variable $c(s) = c(t(s))$.

6–7: The plot, $s(t)$, is monotonically increasing, and therefore the function could always be inverted numerically.

8: Even for the arc-length that could not be evaluated in closed-form (i.e., $\text{PrettyFlower}$), a numerical integration could be used to perform the inversion.

Scalar Functions with Vector Argument

In Materials Science and Engineering, the concept of a spatially varying function arises frequently:

For example:

Concentration $c_i(x, y, z) = c_i(\vec{x})$ is the number (or moles) of chemical species of type $i$ per unit volume located at the point $\vec{x}$.

Density $\rho(x, y, z) = \rho(\vec{x})$ mass per unit volume located at the point $\vec{x}$ is $\rho(x, y, z) = \rho(\vec{x})$.

Energy Density $u(x, y, z) = u(\vec{x})$ energy per unit volume located at the point $\vec{x}$.

The examples above are spatially-dependent densities of “extensive quantities.”
There are also spatially variable intensive quantities:

**Temperature** $T(x, y, z) = T(\vec{x})$ is the temperature which would be measured at the point $\vec{x}$.

**Pressure** $P(x, y, z) = P(\vec{x})$ is the pressure which would be measured at the point $\vec{x}$.

**Chemical Potential** $\mu_i(x, y, z) = \mu_i(\vec{x})$ is the chemical potential of the species $i$ which would be measured at the point $\vec{x}$.

Each example is a scalar function of space—that is, the function associates a scalar with each point in space.

A topographical map is a familiar example of a graphical illustration of a scalar function (altitude) as a function of location (latitude and longitude).

### How Confusion Can Develop in Thermodynamics

However, there are many other types of scalar functions of several arguments, such as the state function: $U = U(S, V, N_i)$ or $P = P(V, T, N_i)$. It is sometimes useful to think of these types of functions a scalar functions of a “point” in a thermodynamics space.

However, this is often a source of confusion: notice that the internal energy is used in two different contexts above. One context is the value of the energy, say 128.2 Joules. The other context is the function $U(S, V, N_i)$. While the two symbols are identical, their meanings are quite different.

The root of the confusion lurks in the question, “What are the variables of $U$?” Suppose that there is only one (independent) chemical species, then $U(\cdot)$ has three variables, such as $U(S, V, N_i)$. “But if $S(T, P, \mu)$, $V(T, P, \mu)$, and $N(T, P, \mu)$ are known functions, then what are the variables of $U$?” The answer is that they are any three independent variables, one could write $U(T, P, \mu) = U(S(T, P, \mu), V(T, P, \mu), N(T, P, \mu))$, and there are six other natural choices.

The trouble arises when variations of a function like $U$ are queried—then the variables that are varying must be specified.

For this reason, it is either a good idea to keep the functional form explicit in thermodynamics, i.e.,

$$
\begin{aligned}
dU(S, V, N) &= \frac{\partial U(S, V, N)}{\partial S} dS + \frac{\partial U(S, V, N)}{\partial V} dV + \frac{\partial U(S, V, N)}{\partial N} dN \\
dU(T, P, \mu) &= \frac{\partial U(T, P, \mu)}{\partial T} dT + \frac{\partial U(T, P, \mu)}{\partial V} dV + \frac{\partial U(T, P, \mu)}{\partial \mu} d\mu
\end{aligned}
$$

or use, the common thermodynamic notation,

$$
\begin{aligned}
dU &= \left(\frac{\partial U}{\partial S}\right)_{V,N} dS + \left(\frac{\partial U}{\partial V}\right)_{S,N} dV + \left(\frac{\partial U}{\partial N}\right)_{S,V} dN \\
dU &= \left(\frac{\partial U}{\partial T}\right)_{P,\mu} dT + \left(\frac{\partial U}{\partial P}\right)_{T,\mu} dP + \left(\frac{\partial U}{\partial \mu}\right)_{T,P} d\mu
\end{aligned}
$$

### Total and Partial Derivatives, Chain Rule

There is no doubt that a great deal confusion arises from the following question, “What are the variables of my function?”

For example, suppose we have a three-dimensional space $(x, y, z)$, in which there is an embedded surface $(x(w, v), y(w, v), z(w, v)) = \vec{x}(\vec{u})$ where $\vec{u} = (v, w)$ is a vector that lies in the surface, and an embedded curve $(x(s), y(s), z(s)) = \vec{x}(s)$. Furthermore, suppose there is a curve that lies within the surface $(w(t), v(t)) = \vec{u}(t)$.

Suppose that $E = f(x, y, z)$ is a scalar function of $(x, y, z)$.

Here are some questions that arise in different applications:
1. How does $E$ vary as a function of position?
2. How does $E$ vary along the surface?
3. How does $E$ vary along the curve?
4. How does $E$ vary along the curve embedded in the surface?
Lecture 12  Mathemtica® Example 3  
Total Derivatives and Partial Derivatives: A Mathematica Review

Demonstrations of 1) the three spatial derivatives of $F(x, y, z)$; 2) the two independent derivatives on a two-dimensional surface embedded in $x-y-z$; 3) the complete derivative of $F(x, y, z)$ along a curve $(x(t), y(t), z(t))$.

3-2: AScalarFunction is a symbolic representation of a function—it will be a place-holder for examples of partial derivatives.

3: This will print Mathematica's representation of derivatives with respect to one of several arguments—e.g., $\frac{\partial F(x, y, z)}{\partial y}$ is written as $F^{(0,1,0)}[x,y,z]$.  

4: AScalarFunction becomes a function of two variables when $x$, $y$, and $z$ are restricted to a surface parameterized by $(u, v)$: $(x(w, u), y(w, v), z(w, v))$.  

6: Caution: the distinction between the symbol $x$ and the symbol $x[w,v]$ is important; the following two examples show how the derivatives should appear.

7: This and the previous example show how the chain rule is computed, these two terms are the components of the gradient in the surface.

8: In this case, the previous AScalarFunction becomes a function of a single variable by specifying a curve in the surface with $(w(t), v(t))$.  

9: Now, a total derivative of the curve embedded in the surface can be calculated with the chain rule. This is nearly equivalent to the following along a specified curve.

10: Here is the total derivative along a specific curve $(x(t), y(t), z(t))$, but here there is no constraint to the surface $\vec{x}(w, v)$

Taylor Series

The behavior of a function near a point is something that arises frequently in physical models. When the function has lower-order continuous partial derivatives (generally, a "smooth" function near the point in question), the stock method to model local behavior is Taylor’s series expansions around a fixed point.

Taylor’s expansion for a scalar function of $n$ variables, $f(x_1, x_2, \ldots, x_n)$, which has continuous first
and second partial derivatives near the point $\tilde{\xi} = (\xi_1, \ldots, \xi_n)$, is:

$$f(\xi_1, \ldots, \xi_n) = f(x_1, x_2, \ldots, x_n) + \frac{\partial f}{\partial x_1}(\xi_1 - x_1) + \frac{\partial f}{\partial x_2}(\xi_2 - x_2) + \ldots + \frac{\partial f}{\partial x_n}(\xi_n - x_n)$$

$$+ \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x_1^2}(\xi_1 - x_1)^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\xi_1 - x_1)(\xi_2 - x_2) + \ldots + \frac{\partial^2 f}{\partial x_1 \partial x_n}(\xi_1 - x_1)(\xi_n - x_n) \right. \right.$

$$\left. + \frac{\partial^2 f}{\partial x_2^2}(\xi_2 - x_2)^2 + \frac{\partial^2 f}{\partial x_2 \partial x_1}(\xi_2 - x_2)(\xi_1 - x_1) + \ldots + \frac{\partial^2 f}{\partial x_2 \partial x_n}(\xi_2 - x_2)(\xi_n - x_n) \right)$$

$$\left. + \ldots \right] + O([\xi_1 - x_1]^3) + O([\xi_1 - x_1]^2(\xi_2 - x_2)] + O([(\xi_1 - x_1)(\xi_2 - x_2)^2] + O[(\xi_2 - x_2)^3]$$

$$+ \ldots + O([(\xi_1 - x_1)^2(\xi_n - x_n)] + O([(\xi_1 - x_1)(\xi_2 - x_2)(\xi_n - x_n)] + \ldots + O[(\xi_n - x_n)^3]$$

or in a vector shorthand:

$$f(\tilde{x}) = f(\tilde{\xi}) + \nabla_{\tilde{x}} f|_{\tilde{\xi}} \cdot (\tilde{\xi} - \tilde{x}) + (\tilde{\xi} - \tilde{x}) \cdot (\nabla_{\tilde{x}} \nabla_{\tilde{x}} f)|_{\tilde{\xi}} \cdot (\tilde{x} - \tilde{x}) + O[\|\tilde{x} - \tilde{x}\|^3]$$

(12-9)

In the following example, visualization of local approximations will be obtained for a scalar function of two variables, $f(x,y)$. This will be extended into an approximating function of four variables by expanding it about a point $(\xi, \eta)$ to second order. The expansion is now a function of four variables—the first two variables are the point the function is expanded around $(x$ and $y)$, and the second two are the variable of the parabolic approximation at that point $(\xi$ and $\eta)$: $f_{appx}(\xi, \eta; x, y) = f(x, y) + \frac{\partial f}{\partial x}|_{x,y}(\xi - x) + \frac{\partial f}{\partial y}|_{x,y}(\eta - y) + Q$ where $Q \equiv \frac{1}{2} \frac{\partial^2 f}{\partial x^2}|_{x,y}(\xi - x)^2 + \frac{\partial^2 f}{\partial x \partial y}|_{x,y}(\xi - x)(\eta - y)$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial y^2}|_{x,y}(\eta - y)^2$$

or $f_{appx}(\xi, \eta; x, y) = f(x, y) + \nabla f \cdot \left( \begin{array}{c} \xi - x \\ \eta - y \end{array} \right) + \frac{1}{2} Q_{\text{form}}$ where $Q_{\text{form}} \equiv (\xi - x, \eta - y) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}|_{x,y} & \frac{\partial^2 f}{\partial x \partial y}|_{x,y} \\ \frac{\partial^2 f}{\partial y \partial x}|_{x,y} & \frac{\partial^2 f}{\partial y^2}|_{x,y} \end{pmatrix} \begin{pmatrix} \xi - x \\ \eta - y \end{pmatrix}$

The function $f_{appx}(\xi, \eta; x, y)$ will be plotted as a function of $\xi$ and $\eta$ for $|\xi - x| < \delta$ and $|\eta - y| < \delta$ for a selected number of points $(x,y)$. 

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This will demonstrate how to produce an expression-form for a first-order expansion to a function of three variables.

1: Here, we get a symbolic representation of the approximation of $F(x + dx, y + dy, z + dz)$ near the point $(x, y, z) = (0, 0, 0)$ to first order. The Series function first expands about the last iterator-argument. This produces a result that multiplies the expansion parameters $dx$, $dy$, and $dz$ by the order of approximation functions $O[dx^2]$.

2: Using Normal and Expand on the series-result above yields an expression for the approximation, but it is not first-order as we might have intended. The $dx \ dy$-terms are a result of expanding the result of the three sequential expansions in the first command.

3: Here, a rule is applied to remove the second-order terms. The result has the same form as the thermodynamic expression:

$$dG = \left( \frac{\partial G}{\partial P} \right)_{T,n} dP + \left( \frac{\partial G}{\partial T} \right)_{n,P} dT + \left( \frac{\partial G}{\partial n} \right)_{P,T} dn$$

$$= V(P, T, n) dP - S(P, T, n) dT + \mu(P, T, n) dn$$
Approximating Surfaces at Points

Visualization of quadratic approximations to a surface at points on that surface

```
CrazyFun[x_, y_] :=
  Sin[5 p x]/(x y) + Sin[5 p (x - 1)]
  Sin[5 p (y - 1)]/(x (y - 5))

theplot = Plot3D[CrazyFun[x, y],
  {x, 0.1, 9}, {y, 0.1, 9}, PlotRange -> All, Mesh -> False, PlotStyle -> {Opacity(0.3)}]

ApproxPlot[x_, y_, xo, yo] :=
  Series[CrazyFun[x, y],
  {x, xo, 2}, {y, yo, 2}]] // Normal

approx = Plot3D[
  Evaluate[ApproxFunction(x, y, xo, yo)],
  {x, xo - 1, xo + 1}, {y, yo - 1, yo + 1}]

Show[theplot, approx,]
```

Just a few of many examples of instances where Taylor’s expansions are used are:

**linearization** Examining the behavior of a model near a point where the model is understood. Even if the model is wildly non-linear, a useful approximation is to make it linear by evaluating near a fixed point.

**approximation** If a model has a complicated representation in terms of unfamiliar functions, a Taylor expansion can be used to characterize the ‘local’ model in terms of a simple polynomial.

**asymptotics** Even when a system has singular behavior (e.g., the value of a function becomes infinite as some variable approaches a particular value), how the system becomes singular is very important. At singular points, the Taylor expansion will have leading order terms that are singular, for example near $x = 0$,

$$\frac{\sin(x)}{x^2} = \frac{1}{x} - \frac{x}{6} + \mathcal{O}(x^3) \quad (12-10)$$

The singularity can be subtracted off and it can be said that this function approaches $\infty$ "linearly" from below with slope $-1/6$. Comparing this to the behavior of another function that is singular near zero:

$$\frac{\exp(x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + O(x^3)$$  \hfill (12-11)

shows that the $e^x/x$ behavior is "more singular."

```
Plot[{{Sin[x]/x^2 - 1/x, Exp[x]/x - 1/x}, {x, .001, 2.5}},
PlotStyle -> {{Thickness[0.02], Hue[1]}, {Thickness[0.01], Hue[0.5]}}]
```

![Figure 12-10: Behavior of two singular functions near their singular points.](image)

**Stability** If the expansion of energy about a point is obtained, then the various orders of expansion can be interpreted:

**Zero-order** The zeroth-order term in a local expansion is the energy of the system at the point evaluated. Unless this term is to be compared to another point, it has no particular meaning (if it is not infinite), as energy is always arbitrarily defined up to a constant.

**First-order** The first-order is related to the driving force to change the state of the system. Consider:

$$\Delta E = \nabla E \cdot \delta \vec{x} = -\vec{F} \cdot \delta \vec{x}$$  \hfill (12-12)

If force exists, the system can decrease its energy linearly by picking a particular change $\delta \vec{x}$ that is anti-parallel to the force.

For a system to be stable, it is a necessary first condition that the forces (or first order expansion coefficients) vanish.
**second order** If a system has no forces on it (therefore satisfying the necessary condition of stability), then where the system is stable or unstable depends on whether a small \( \delta \vec{x} \) can be found that decreases the energy:

\[
\Delta E = \frac{1}{2} \delta \vec{x} \cdot \nabla(-\vec{F}) \cdot \delta \vec{x} \\
= \frac{1}{2} \cdot \nabla \nabla E \cdot \delta \vec{x} \\
= \frac{1}{2} \left. \frac{\partial^2 E}{\partial x_i \partial x_j} \right|_{x_1, x_2, \ldots, x_n} \delta x_i \delta x_j
\]

where the summation convention is used above and the point \((x_1, x_2, \ldots, x_n)\) is one for which \( \nabla E \) is zero.

**numerics** Derivatives are often obtained numerically in numerical simulations. The Taylor expansion provides a formula to approximate numerical derivatives—and provides an estimate of the numerical error as a function of quantities like numerical mesh size.

### Gradients and Directional Derivatives

Scalar functions \( F(x, y, z) = F(\vec{x}) \) have a natural vector field associated with them—at each point \( \vec{x} \) there is a direction \( \hat{n}(\vec{x}) \) pointing in the direction of the most rapid increase of \( F \). Associating the **magnitude** of a vector in the direction of steepest increase with the **rate of increase** of \( F \) defines the gradient.

The gradient is therefore a vector function with a vector argument (\( \vec{x} \) in this case) and it is commonly written as \( \nabla F \).

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Here are some natural examples:

**Meteorology** The “high pressure regions” are commonly displayed with weather reports—as are the "isobars" or curves of constant barometric pressure. Thus displayed, pressure is a scalar function of latitude and longitude.

At any point on the map, there is a direction that points to a local high pressure center—this is the direction of the gradient. The rate at which the pressure is increasing gives the magnitude of the gradient.

The gradient of pressure should be a vector that is related to the direction and the speed of wind.

**Mosquitoes** It is known that hungry mosquitoes tend to fly towards sources (or local maxima) of carbon dioxide. Therefore, it can be hypothesized that mosquitoes are able to determine the gradient in the concentration of carbon dioxide.

**Heat** In an isolated system, heat flows from high-temperature \((T(\vec{x}))\) regions to low-temperature regions.
The Fourier empirical law of heat flow states that the rate of heat flows is proportional to the local decrease in temperature.

Therefore, the local rate of heat flow should be proportional to the vector which is minus the gradient of $T(\vec{x})$: $-\nabla T$

Finding the Gradient

Potentials and Force Fields

Force is a vector. Force projected onto a displacement vector $d\vec{x}$ is the rate at which work, $dW$, is done on an object $dW = -\vec{F} \cdot d\vec{x}$.

If the work is being supplied by an external agent (e.g., a charged sphere, a black hole, a magnet, etc.), then it may be possible to ascribe a potential energy ($E(\vec{x})$, a scalar function with vector argument) to the agent associated with the position at which the force is being applied. This $E(\vec{x})$ is the potential for the agent and the force field due to the agent is $\vec{F}(\vec{x}) = -\nabla E(\vec{x})$.

Sometimes the force (and therefore the energy) scales with the “size” of the object (i.e., the object’s total charge in an electric potential due to a fixed set of charges, the mass of an object in the gravitational potential of a black hole, the magnetization of the object in a magnetic potential, etc.). In these cases, the potential field can be defined in terms of a unit size (per unit charge, per unit mass, etc.). One can determine whether such a scaling is applied by checking the units.

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5 As with any energy, there is always an arbitrary constant associated with the position (or state) at which the energy is taken to be zero. There is no such ambiguity with force. Forces are, in a sense, more fundamental than energies. Energy appears to be fundamental because all observations of the first law of thermodynamics demonstrate that there is a conserved quantity which is a state function and is called energy.