Lecture 10: Real Eigenvalue Systems; Transformations to Eigenbasis

Reading:
Kreyszig Sections: 8.4, 8.5 (pages349–354, 356–361)

Similarity Transformations

A matrix has been discussed as a linear operation on vectors. The matrix itself is defined in terms of the coordinate system of the vectors that it operates on—and that of the vectors it returns.

In many applications, the coordinate system (or laboratory) frame of the vector that gets operated on is the same as the vector gets returned. This is the case for almost all physical properties. For example:

- In an electronical conductor, local current density, $\vec{J}$, is linearly related to the local electric field $\vec{E}$:

$$\underline{\rho}\vec{J} = \vec{E} \tag{10-1}$$

- In a thermal conductor, local heat current density is linearly related to the gradient in temperature:

$$-\underline{k}\nabla T = \vec{j_Q} \tag{10-2}$$

- In diamagnetic and paramagnetic materials, the local magnetization, $\vec{B}$ is related to the applied field, $\vec{H}$:

$$\underline{\mu}\vec{H} = \vec{B} \tag{10-3}$$

- In dielectric materials, the local total polarization, $\vec{D}$, is related to the applied electric field:

$$\underline{\kappa}\vec{E} = \vec{D} = \kappa_o \vec{E} + \vec{P} \qquad (10\text{-}4)$$

When $\vec{x}$ and $\vec{y}$ are vectors representing a physical quantity in Cartesian space (such as force $\vec{F}$, electric field $\vec{E}$, orientation of a plane $\hat{n}$, current $\vec{j}$, etc.) they represent something *physical*. They don't change if we decide to use a different space in which to represent them (such as, exchanging x for y, y for z, z for x; or, if we decide to represent length in nanometers instead of inches, or if we simply decide to rotate the system that describes the vectors. The representation of the vectors themselves may change, but they stand for the same thing.

One interpretation of the operation $\underline{A}\vec{x}$ has been described as geometric transformation on the vector $\vec{x}$. For the case of orthogonal matrices $\underline{A}_{orth}$, geometrical transformations take the forms of rotation, reflection, and/or inversion.

Suppose we have some *physical* relation between two physical vectors in some coordinate system, for instance, the general form of Ohm's law is:

$$\vec{J} = \underline{\chi}\vec{E}$$

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{xy} & \chi_{yy} & \chi_{yz} \\ \chi_{xz} & \chi_{yz} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (10\text{-}5)$$

The matrix (actually it is better to call it a *rank-2 tensor*) $\underline{\chi}$ is a physical quantity relating the amount of current that flows (in a direction) proportional to the applied electric field (perhaps in a different direction). $\underline{\chi}$ is the "conductivity tensor" for a particular material.

The physical law in Eq. 10-5 can be expressed as an inverse relationship:

$$\vec{E} = \underline{\rho}\vec{j}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & \rho_{yy} & \rho_{yz} \\ \rho_{xz} & \rho_{yz} & \rho_{zz} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} \qquad (10\text{-}6)$$

where the resistivity tensor $\underline{\rho} = \underline{\chi}^{-1}$.

What happens if we decide to use a new coordinate system (i.e., one that is rotated, reflected, or inverted) to describe the relationship expressed by Ohm's law?

The two vectors must transform from the "old" to the "new" coordinates by:

$$\underline{A_{orth}^{old \to new}} \vec{E^{old}} = \vec{E^{new}} \qquad \underline{A_{orth}^{old \to new}} \vec{j^{old}} = \vec{j^{new}}$$

$$\underline{A_{orth}^{new \to old}} \vec{E^{new}} = \vec{E^{old}} \qquad \underline{A_{orth}^{new \to old}} \vec{j^{new}} = \vec{j^{old}} \tag{10-7}$$

Where is simple proof will show that:

$$\underline{A_{orth}^{old \to new}} = \underline{A_{orth}^{new \to old}}^{-1}$$

$$\underline{A_{orth}^{new \to old}} = \underline{A_{orth}^{old \to new}}^{-1}$$

$$\underline{A_{orth}^{new \to old}} = \underline{A_{orth}^{old \to new}}^{T} \tag{10-8}$$

$$\underline{A_{orth}^{new \to old}} = \underline{A_{orth}^{old \to new}}^{T}$$

where the last two relations follow from the special properties of orthogonal matrices.

How does the physical law expressed by Eq. 10-5 change in a new coordinate system?

$$\text{in old coordinate system: } \vec{j^{old}} = \underline{\chi^{old}} \vec{E^{old}}$$

$$\text{in new coordinate system: } \vec{j^{new}} = \underline{\chi^{new}} \vec{E^{new}} \tag{10-9}$$

To find the relationship between $\underline{\chi^{old}}$ and $\underline{\chi^{new}}$: For the first equation in 10-9, using the transformations in Eqs. 10-7:

$$\underline{A_{orth}^{new \to old}} \vec{j^{new}} = \underline{\chi^{old}} \underline{A_{orth}^{new \to old}} \vec{E^{new}} \tag{10-10}$$

and for the second equation in 10-9:

$$\underline{A_{orth}^{old \to new}} \vec{j^{old}} = \underline{\chi^{new}} \underline{A_{orth}^{old \to new}} \vec{E^{old}} \tag{10-11}$$

Left-multiplying by the inverse orthogonal transformations:

$$\underline{A_{orth}^{old \to new}} \, \underline{A_{orth}^{new \to old}} \vec{j^{new}} = \underline{A_{orth}^{old \to new}} \, \underline{\chi^{old}} \, \underline{A_{orth}^{new \to old}} \vec{E^{new}}$$
$$\underline{A_{orth}^{new \to old}} \, \underline{A_{orth}^{old \to new}} \vec{j^{old}} = \underline{A_{orth}^{new \to old}} \, \underline{\chi^{new}} \, \underline{A_{orth}^{old \to new}} \vec{E^{old}} \tag{10-12}$$

Because the transformation matrices are inverses, the following relationship between *similar* matrices in the old and new coordinate systems is:

$$\underline{\chi^{old}} = \underline{A_{orth}^{old \to new}} \, \underline{\chi^{new}} \, \underline{A_{orth}^{new \to old}}$$
$$\underline{\chi^{new}} = \underline{A_{orth}^{new \to old}} \, \underline{\chi^{old}} \, \underline{A_{orth}^{old \to new}} \tag{10-13}$$

The $\underline{\chi^{old}}$ is said to be *similar* to $\underline{\chi^{new}}$ and the double multiplication operation in Eq. 10-13 is called *a similarity transformation*.

Stresses and Strains

Stresses and strains are rank-2 tensors that characterize the mechanical state of a material.

A spring is an example of a one-dimensional material—it resists or exerts force in one direction only. A volume of material can exert forces in all three directions simultaneously—and the forces need not be the same in all directions. A volume of material can also be "squeezed" in many different ways: it can be squeezed along any one of the axis or it can be subjected to squeezing (*or smeared*) around any of the axes[3]

All the ways that a force can be applied to small element of material are now described. A force divided by an area is a *stress*—think of it the area density of force.

$$\sigma_{ij} = \frac{F_i}{A_j} \qquad (\text{i.e., } \sigma_{xz} = \frac{F_x}{A_z} = \sigma_{xz} = \frac{\vec{F} \cdot \hat{i}}{\vec{A} \cdot \hat{k}}) \tag{10-14}$$

A_j is a plane with its normal in the $\hat{j}$-direction (or the projection of the area of a plane $\vec{A}$ in the direction parallel to $\hat{j}$)

---

[3]Consider a blob of modeling clay—you can deform it by placing between your thumbs and one opposed finger; you can deform it by simultaneously squeezing with two sets of opposable digits; you can "smear" it by pushing and pulling in opposite directions. These are examples of uniaxial, biaxial, and shear stress.
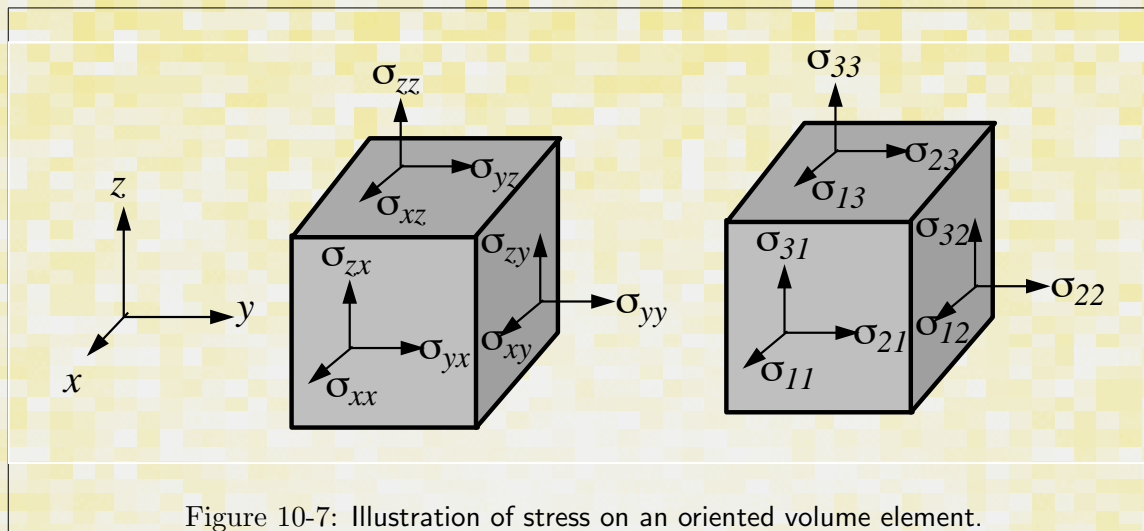
Figure 10-7: Illustration of stress on an oriented volume element.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (10\text{-}15)$$

There is one special and very simple case of elastic stress, and that is called the hydrostatic stress. It is the case of pure pressure and there are no shear (off-diagonal) stresses (i.e., all $\sigma_{ij} = 0$ for $i \neq j$, and $\sigma_{11} = \sigma_{22} = \sigma_{33}$). An equilibrium system composed of a body in a fluid environment is always in hydrostatic stress:

$$\sigma_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} \qquad (10\text{-}16)$$

where the pure hydrostatic pressure is given by P.

Strain is also a rank-2 tensor and it is a physical measure of a how much a material changes its shape.[4]

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[4]It is unfortunate that the words of these two related physical quantities, stress and strain, sound so similar. Strain measures the change in
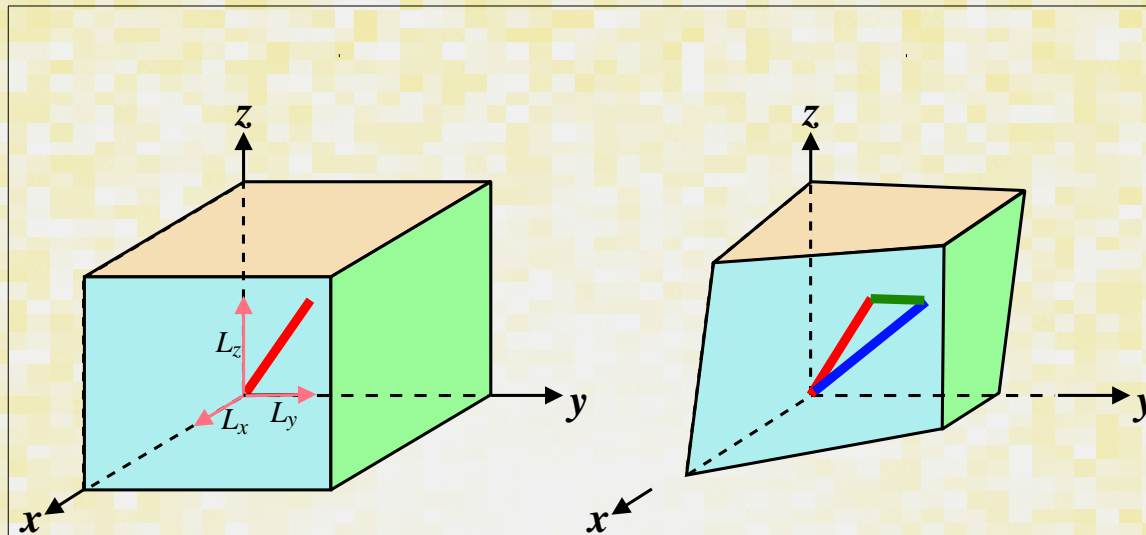
Why should strain be a rank-2 tensor?

Figure 10-8: Illustration of how strain is defined: imagine a small line-segment that is aligned with a particular direction (one set of indices for the direction of the line-segment); after deformation the end-points of the line segment define a new line-segment in the deformed state. The difference in these two vectors is a vector representing how the line segment has changed from the initial state into the deformed state. The difference vector can be oriented in any direction (the second set of indices)—the strain is a representation of "a difference vectors for all the oriented line-segments" divided by the length of the original line.

Or, using the same idea as that for stress:

$$\epsilon_{ij} = \frac{\Delta L_i}{L_j} \qquad (\text{i.e., } \epsilon_{xz} = \frac{\Delta L_x}{L_z} = \epsilon_{xz} = \frac{\vec{\Delta L} \cdot \hat{i}}{\vec{L} \cdot \hat{k}}) \qquad (10\text{-}17)$$

geometry of a body and stress measures the forces that squeeze or pull on a body. St**ress** is the p**ress**; St**rain** is the g**ain**.

If a body that is being stressed hydro-statically is isotropic, then its response is pure dilation (in other words, it expands or shrinks uniformly and without shear):

$$\epsilon_{ij} = \begin{bmatrix} \Delta/3 & 0 & 0 \\ 0 & \Delta/3 & 0 \\ 0 & 0 & \Delta/3 \end{bmatrix} \tag{10-18}$$

$$\Delta = \frac{dV}{V} \tag{10-19}$$

So, for the case of hydrostatic stress, the work term has a particularly simple form:

$$V \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} d\epsilon_{ij} = -PdV$$

$$V \sigma_{ij} d\epsilon_{ij} = -PdV \qquad \text{(summation convention)} \tag{10-20}$$

This expression is the same as the rate of work performed on a compressible fluid, such as an ideal gas.

EigenStrains and EigenStresses

For any strain matrix, there is a choice of an coordinate system where line-segments that lie along the coordinate axes always deform parallel to themselves (i.e., they only stretch or shrink, they do not twist).

For any stress matrix, there is a choice of an coordinate system where all shear stresses (the off-diagonal terms) vanish and the matrix is diagonal.

These coordinate systems define the eigenstrain and eigenstress. The matrix transformation that takes a coordinate system into its eigenstate is of great interest because it simplifies the mathematical representation of the physical system.

Representations of Stress (or Strain) in Rotated Coordinate Systems

A demonstration of rotating a quasi-two dimensional stress state is given. Convenient forms for the stress in any coordinate system are derived. The stress invariants are demonstrated.

This is a general state, we will rotate about the z-axis and compare the result to a general two-dimensional stress state.

σtensordiag = $\begin{pmatrix} \sigma\text{princ}_{xx} & 0 & 0 \\ 0 & \sigma\text{princ}_{yy} & 0 \\ 0 & 0 & \sigma\text{princ}_{zz} \end{pmatrix}$;

σtensordiag // MatrixForm

1

rotmat[θ_] := $\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] & 0 \\ \text{Sin}[\theta] & \text{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

rotmat[θ] // MatrixForm

2

Transformation to general two-dimensional stress state coordinate system by rotating the principal system by θ around z-axis

σrot = Simplify[Transpose[rotmat[θ]].
     σtensordiag.rotmat[θ]];

σrot // MatrixForm

3

Writing the same equation in a slightly different way...

σrotalt = Collect[
     σrot // TrigReduce, {Cos[2 θ], Sin[2 θ]}];

σrotalt // MatrixForm

4

Naming the coefficients of the rotated two-dimensional state:

σlabMat = $\begin{pmatrix} \sigma\text{lab}_{xx} & \sigma\text{lab}_{xy} & \sigma\text{lab}_{xz} \\ \sigma\text{lab}_{xy} & \sigma\text{lab}_{yy} & \sigma\text{lab}_{yz} \\ \sigma\text{lab}_{xz} & \sigma\text{lab}_{yz} & \sigma\text{lab}_{zz} \end{pmatrix}$ = σrotalt;

σlabMat // MatrixForm

5

Looking at the x-y components of stress (i.e, the upper-left 2×2 submatrix), notice that there are two invariants of the generalized two-dimensional stress state: The trace and the determinant:

Simplify[σlab$_{xx}$ + σlab$_{yy}$]

6

Simplify$\left[\sigma\text{lab}_{xx}\ \sigma\text{lab}_{yy} - \left(\sigma\text{lab}_{xy}\right)\text{^2}\right]$

7

Do not depend on θ; thus illustrating the invariance of these quantities under rotation of coordinate rotations.

1: The problem is done in reverse by finding the backwards rotation of a diagonal matrix. This is the stress in the principle coordinate system (it is diagonalized with eigenvalue entries) Any rotation similarity transformation on this matrix is the equivalent stress in the rotated frame.

2: This is the rotation operator by counter-clockwise angle θ about the z-axis.

3: Therefore σrot, obtained by the similarity transformation, is the stress in any rotation about the z-axis. It is a quasi-two dimensional state defined by $\sigma_{zx} = \sigma_{yz} = 0$.

4: The rotation matrix factors well using the double angle formulas; here we use **TrigReduce** to convert powers of trigonometric functions into functions of multiples of their angle, and the **Collect** the double-angle terms.

5: This can be compared with the general form of any quasi-two-dimensional (x–y plane) stress that has the same principle stresses identified above. In the rotated (i.e., laboratory) frame, the stresses are geometrically related to the circle plotted in 10-9, which will be also visualized in the following examples.

6: Here, we use double assignment for the laboratory-frame matrix and, simultaneously, its elements.

7: This will show that the trace of the stress tensor is independent of rotation—this is a general property for any unitary transformation.

8: Like the trace, the determinant is also an *matrix invariant*.

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Principal Axes: Mohr's Circle of Two-Dimensional Stress

By diagonalizing a quasi-two-dimensional stress tensor, the equations for Mohr's circle of stress (Fig. 10-9) are derived.

1. σlab_{xx} in laboratory system rotated by θ from principal axis system

```
σlab_xx                                              1
```

2. σlab_{yy} in laboratory system rotated by θ from principal axis system

```
σlab_yy                                              2
```

3. σlab_{xy} in laboratory system rotated by θ from principal axis system

```
σlab_xy                                              3
```

```
uniaxial10 = {σprinc_xx -> 10, σprinc_yy -> 0}       4
```

```
ParametricPlot[{σlab_xx, σlab_xy} /. uniaxial10
, {θ, 0, π}, AxesLabel →
  {"normal stress", "shear stress"},
AspectRatio → 1, PlotLabel → " \t \t Mohr       5
   Circle for 10 MPa Uniaxial Tension",
PlotStyle → {Thickness[0.01], Hue[1]}]
```

```
uniaxialother = {σprinc_xx -> 30, σprinc_yy -> 10}   6
```

```
ParametricPlot[
{σlab_xx, σlab_xy} /. uniaxialother
, {θ, 0, π}, AxesLabel → {"normal stress",
  "shear stress"}, AspectRatio → 1,
PlotRange -> {{0, 40}, {-20, 20}},               7
PlotLabel → " \t \t Mohr Circle
   for σprinc_xx= 30   σprinc_yy=10",
PlotStyle → {Thickness[0.01], Hue[1]}]
```

1–3: These are the forms of the three two-dimensional stress components are simple expressions in terms of 2θ. We will see how these equations produce a circle.

4: This is a rule that defines *a particular* stress state in terms of the principal stresses (here given by 10 and 0).

5: 5–7 Using `ParametricPlot` for $\sigma_{xx}(\theta)$ and $\sigma_{xy}(\theta)$, an example of Mohr's circle is plotted for examples of principle stress for (0,10) and (30,10). The interpretation of this figure is given in Fig. 10-9.
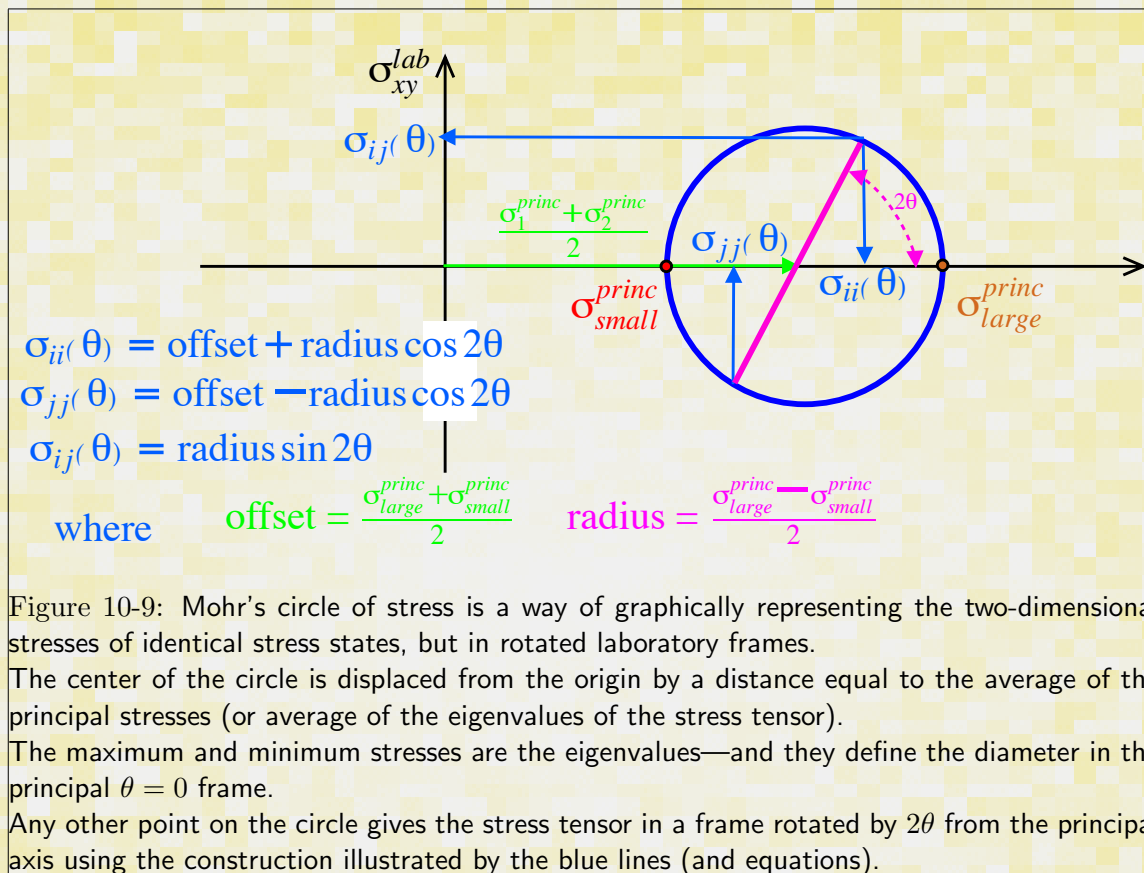
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$$\sigma_{ii}(\theta) = \text{offset} + \text{radius} \cos 2\theta$$
$$\sigma_{jj}(\theta) = \text{offset} - \text{radius} \cos 2\theta$$
$$\sigma_{ij}(\theta) = \text{radius} \sin 2\theta$$

where $\quad \text{offset} = \dfrac{\sigma_{large}^{princ} + \sigma_{small}^{princ}}{2} \quad \text{radius} = \dfrac{\sigma_{large}^{princ} - \sigma_{small}^{princ}}{2}$

Figure 10-9: Mohr's circle of stress is a way of graphically representing the two-dimensional stresses of identical stress states, but in rotated laboratory frames.
The center of the circle is displaced from the origin by a distance equal to the average of the principal stresses (or average of the eigenvalues of the stress tensor).
The maximum and minimum stresses are the eigenvalues—and they define the diameter in the principal $\theta = 0$ frame.
Any other point on the circle gives the stress tensor in a frame rotated by 2θ from the principal axis using the construction illustrated by the blue lines (and equations).

Visualization Example: Graphics for Mohr's Circle

Our goal is to produce a manipulatable and interactive graphical representation for Mohr's circle of stress. We break the problem up by creating the individual graphical elements that appear in Fig. 10-9. These functions will be utilized in the following example.

```
mohrs[off_, rad_] :=
 {Red, Thick, Circle[{offset, 0}, radius]}
```
1

```
s12graph[s11_, s12_] := {Darker[Orange],
  Arrow[{{s11, s12}, {0, s12}}],
  Text[s12, {0, s12},
   {-1.5, -1.5}, Background → White]}
```
2

```
s22graph[s22_, s12_] :=
 {Blue, Arrow[{{s22, -s12}, {s22, 0}}],
  Text[s22, {s22, 0}, {0, -1},
   Background → White]}
```
3

```
s11graph[s11_, s12_] := {Darker[Green],
  Dynamic[Arrow[{{s11, s12}, {s11, 0}}]],
  Dynamic[Text[s11, {s11, 0},
   {0, 1}, Background → White]]}
```
4

```
diamgraph[s11_, s12_, s22_] :=
 {Line[{{s22, -s12}, {s11, s12}}]}
```
5

```
anglegraph[twotheta_] :=
 {Purple, Dashed, Circle[{offset, 0},
  1.2 * radius, {0, twotheta}], Text[
  Style["2θ=" <> ToString[twotheta 180 / Pi],
   Medium], {offset, 0} + 1.2 * radius *
   {Cos[twotheta / 2], Sin[twotheta / 2]},
  Background → White]}
```
6

```
titlegraph[s11_, s12_, s22_] :=
 Text[MatrixForm[
  {{Text[Style[s11, Darker[Green], Large]],
    Text[Style[s12, Darker[Orange], Large]]},
   {Text[Style[s12, Darker[Orange], Large]],
    Text[Style[s22, Blue, Large]]}}]]
```
7

1: The function *mohrs* will produce *graphical primitives* at an arbitrary `offset`$=(\sigma_{xx} + \sigma_{yy})/2$ and `radius`$==(\sigma_{xx} - \sigma_{yy})/2$.

2: This will be an annotated dark orange horizontal arrow for the shear stress value σ_{xy}

3: This will be an annotated blue vertical arrow for the stress σ_{yy}

4: This will be an annotated dark green vertical arrow for the stress σ_{xx}

5: This will illustrate the diameter of the circle in the current θ-orientation.

6: This will annotate the current angle 2θ.

7: This creates a title for the graph with entries colored corresponding to the arrows on the graph.

Interactive Graphics Demonstration for Mohr's Circle

We create graphics in which we can use the mouse to drag elements that dynamically update the visualization. In this case, we create a handle that will allow visualization of two-dimensional stress in an arbitrary rotation. The dynamic graphics are enclosed within `Manipulate` so that the original stress state can be set by the user.

```
Manipulate[
 offset = (σ₁₁ + σ₂₂) / 2; radius = (σ₁₁ - σ₂₂) / 2;
 ThetaInit = ArcSin[σ₁₂ / radius] / 2;
 DynamicModule[{twotheta = 2 ThetaInit,
    s11 = 3, s12 = 4, s22}, LocatorPane[
   Dynamic[({radius Cos[twotheta] + offset,
       radius Sin[twotheta]}),
    (twotheta = Mod[Apply[ArcTan,
          (# - {offset, 0})], 2 Pi]) &],
   s11 = Dynamic[offset + radius
       Cos[twotheta]];
   s12 = Dynamic[ radius Sin[twotheta]];
   s22 =
    Dynamic[offset - radius Cos[twotheta]];
   Graphics[{mohrs[offset, radius],
     Dynamic[anglegraph[twotheta]],
     Dynamic[diamgraph[s11, s12, s22]],
     Dynamic[s12graph[s11, s12]] ,
     Dynamic[s22graph[s22, s12]] ,
     Dynamic[s11graph[s11, s12]] },
    Axes → True, AxesOrigin → {0, 0},
    PlotLabel →
     Dynamic[titlegraph[s11, s12, s22]]]
   ]
 ], {{σ₁₁ , 8}, -1, 11},
 {{σ₂₂ , 3.0}, -5, 7}, {{σ₁₂ , 2}, -5, 5},
 FrameLabel → Text[
   Style["Mohr's Circle of Stress", Large]]]
```

1

1: `Manipulate` will get sliders for each of σ_{11}, σ_{22}, and σ_{33} (see note below about these values). The circle offset, radius, and the value of 2θ are calculated first from the Mohr's equation formula. Because we will be changing θ, its current value and the stress state for that rotation will change. We make these variables local (as in `Module`), but further specify that they will be dynamically updated by using `DynamicModule`.

We use `LocatorPane` to place a "handle" at a specific spot which can be updated with the mouse. As the value of θ is changed, we must inform the symbols (here `twotheta`, `s11`, `s12`, and `s22`) are to be updated by embedding them in the `Dynamic` function.

The graphics are drawn using the functions from the previous example.

In this example, no precautions are made to ensure that the stress components will be real in every orientation.
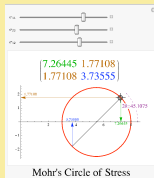
Quadratic Forms

The example above, where a matrix (rank-2 tensor) represents a material property, can be understood with a useful geometrical interpretation.

For the case of the conductivity tensor $\underline{\sigma}$, the dot product $\vec{E} \cdot \vec{j}$ is a scalar related to the local energy dissipation:

$$e = \vec{E}^T \underline{\sigma} \vec{E} \tag{10-21}$$

The term on the right-hand-side is called a quadratic form, as it can be written as:

$$\begin{aligned} e =& \sigma_{11} x_1^2 + \sigma_{12} x_1 x_2 + \sigma_{13} x_1 x_3 + \\ & \sigma_{21} x_1 x_2 + \sigma_{22} x_2^2 + \sigma_{23} x_2 x_3 + \\ & \sigma_{31} x_1 x_3 + \sigma_{32} x_2 x_3 + \sigma_{33} x_3^2 \end{aligned} \tag{10-22}$$

or, because $\underline{\sigma}$ is symmetric:

$$\begin{aligned} e =& \sigma_{11} x_1^2 + 2\sigma_{12} x_1 x_2 + 2\sigma_{13} x_1 x_3 + \\ & \sigma_{22} x_2^2 + 2\sigma_{23} x_2 x_3 + \\ & \sigma_{33} x_3^2 \end{aligned} \tag{10-23}$$

It is not unusual for such quadratic forms to represent energy quantities. For the case of paramagnetic and diamagnetic materials with magnetic permeability tensor $\underline{\mu}$, the energy per unit volume due to an applied magnetic field $\vec{H}$ is:

$$\frac{E}{V} = \frac{1}{2} \vec{H}^T \underline{\mu} \vec{H} \tag{10-24}$$

for a dielectric (i.e., polarizable) material with electric electric permittivity tensor $\underline{\kappa}$ with an applied electric field $\vec{E}$:

$$\frac{E}{V} = \frac{1}{2} \vec{E}^T \underline{\kappa} \vec{E} \tag{10-25}$$

The geometric interpretation of the quadratic forms is obtained by turning the above equations around and asking—what are the general vectors $\vec{x}$ for which the quadratic form (usually an energy or power density) has a particular value? Picking that particular value as unity, the question becomes what are the directions and magnitudes of $\vec{x}$ for which

$$1 = \vec{x}^T \underline{A} \vec{x} \tag{10-26}$$

This equation expresses a quadratic relationship between one component of $\vec{x}$ and the others. This is a surface—known as the *quadric surface* or *representation quadric*—which is an ellipsoid or hyperboloid sheet on which the quadratic form takes on the particular value 1.

In the principal axes (or, equivalently, the eigenbasis) the quadratic form takes the quadratic form takes the simple form:

$$e = \vec{x_{\mathrm{eb}}}^T \underline{A_{\mathrm{eb}}} \vec{x_{\mathrm{eb}}} = A_{11} x_1^2 + A_{22} x_2^2 + A_{33} x_3^2 \tag{10-27}$$

and the representation quadric

$$A_{11} x_1^2 + A_{22} x_2^2 + A_{33} x_3^2 = 1 \tag{10-28}$$

which is easily characterized by the signs of the coefficients.

In other words, in the principal axis system (or the eigenbasis) the quadratic form has a particularly simple, in fact the most simple, form.

Eigenvector Basis

Among all similar matrices (defined by the similarity transformation defined by Eq. 10-13), the simplest matrix is the diagonal one. In the coordinate system where the similar matrix is diagonal, its diagonal entries are the eigenvalues. The question remains, "what is the coordinate transformation that takes the matrix into its diagonal form?"

The coordinate system is called the eigenbasis or principal axis system, and the transformation that takes it there is particularly simple.

The matrix that transforms from a general (old) coordinate system to a diagonalized matrix (in the new coordinate system) is the matrix of columns of the eigenvectors. The first column corresponds to the first eigenvalue on the diagonal matrix, and the nth column is the eigenvector corresponding the nth eigenvalue.

$$\begin{pmatrix} \text{The} \\ \text{Diagonalized} \\ \text{Matrix} \end{pmatrix} = \begin{pmatrix} \text{Eigenvector} \\ \text{Column} \\ \text{Matrix} \end{pmatrix}^{-1} \begin{pmatrix} \text{The} \\ \text{General} \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \text{Eigenvector} \\ \text{Column} \\ \text{Matrix} \end{pmatrix}$$

(10-29)

This method provides a method for finding the simplest quadratic form.

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