Problem Set 4: Out: 19 Oct. and Due: 12 Nov. 2009

Individual assignments should be a combination of your hand-worked solutions and other printed material—they should be placed in the mailbox outside Prof. Carter’s door. Email group assignments to 3016-psets(the symbol at)pruffle.mit.edu

For the individual problems indicated as “Handworked”, you should work your solutions by hand and show your work. Print the results of software-worked solutions, and staple them to your hand-worked assignments before turning them in.

The following are this week’s randomly assigned homework groups. The first member of the group is the “Homework Jefe” who will be in charge of setting up work meetings and have responsibility for turning in the group’s homework notebook. If some some reason, the first member in the list is incapacitated, recalcitrant, or otherwise unavailable, then the second member should take that position. Attention slackers: The Jefe should include a line at the top of your notebook listing the group members that participated in the notebook’s production; only those listed will receive credit. Group names are boldfaced text.

Seymour: alobeidi, arathir, dyyoung, lvegter, gerhardt, poojay, garo
Buddy: jbreucop, p.desai, azook, juquez, joke021, ckleber
Boo-Boo: mmann, agaro, bmiglesi, b.jones, emcisaac, brendapa, pmelo
Walt: mikeyurk, hsi, j.obrien, jcybarra, paraiba, teby
Waker: yasmined, khessler, kcasteel, nathanp, swhudson, mataeux, aparna_s
Zooey: tkish, grahams, soajung,jsteimeI, kparedes, bwee
Franny: phillie, cku313, cklyons, szipparo, sabago, mirnas
Individual (Handworked) Exercise I4-1

Suppose that an amount of water-soluble ink $M_o$ is injected into a very long and thin tube of water at $-\epsilon < x < \epsilon$ (in effect at $x = 0$) and $t = 0$. The equation that governs the diffusion of ink is the diffusion equation, $\partial c/\partial t = D\partial^2 c/\partial x^2$, for this case is

$$c(x, t) = \frac{M_o}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)}$$

1. Show that the above equation solves the diffusion equation.

2. Show that the total mass $M_o$ is conserved for all times $t$.

3. Suppose that the water in the tube is flowing at constant and uniform rate $v_o$ past a fixed observer at $x = 0$. What is the form of the diffusion equation in the observer’s frame of reference?

4. What is the solution to the diffusion equation according to the observer?

5. According to the observer, how does the flux of ink past the observer depend on time?

Individual Exercise I4-2

1. Show that

$$c(x, y, z, t) = \frac{c_o}{8} \times \left[ \text{erf} \left( \frac{x + a/2}{\sqrt{4Dt}} \right) - \text{erf} \left( \frac{x - a/2}{\sqrt{4Dt}} \right) \right] \times \left[ \text{erf} \left( \frac{y + a/2}{\sqrt{4Dt}} \right) - \text{erf} \left( \frac{y - a/2}{\sqrt{4Dt}} \right) \right] \times \left[ \text{erf} \left( \frac{z + a/2}{\sqrt{4Dt}} \right) - \text{erf} \left( \frac{z - a/2}{\sqrt{4Dt}} \right) \right]$$

is a solution to the diffusion equation $\partial c/\partial t = D\nabla^2 c$. Visualize the solution behavior in the plane $z = 0$.

2. Characterize the solution in the limit of very small times, $Dt \ll a^2$.

3. Expand the solution for $Dt \gg a^2$. Write it in terms of $r = \sqrt{x^2 + y^2 + z^2}$ and $t$, and show that it is a solution to the diffusion equation $\partial c/\partial t = D\nabla^2 c$ in spherical coordinates.

4. Characterize and visualize the solution directly above in spherical coordinates.

Individual Exercise I4-3

1. Show that

$$c(x, y, z, t) = c_o - \frac{4c_o}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i \cos \left[ \frac{(2i+1)\pi x}{2L} \right]}{2i + 1} \exp \left[ -\frac{D(2i + 1)^2\pi^2 t}{(2L)^2} \right]$$

is a solution to the diffusion equation on $-L < x < L$. 


2. Characterize the solution at times \( t \ll L^2/t \).

3. Characterize the solution at times \( t \gg L^2/t \).

4. Visualize the solution behavior.

**Individual Exercise I4-4**
Find the Fourier series for this periodic function on on \( 0 < x < \lambda \).

\[
f(x) \quad x = 0 \quad x = \alpha \quad x = \lambda
\]

Visualise your solution for various \( a \) and \( h \).

**Group Exercise G4-1**
The objective of this problem is to compute and visualize the equilibrium state of a simple thermodynamic system. You will need to use the molar latent heat of melting for water for this problem, as well as the molar (constant pressure) heat capacities of ice and liquid-water. You may treat the heat capacities as independent of \( T \), but you can also use a more accurate integration if you wish.

Suppose that you insert water-liquid, and water-ice into an adiabatic container with fixed constant pressure, 1 atm (e.g., a calorimeter).

The inserted liquid is initially at \( 0^\circ C \leq T_{\text{init}}^{\text{liq}} \leq 100^\circ C \) and 1 atm.; the inserted ice is initially at \(-273^\circ C \leq T_{\text{init}}^{\text{sol}} \leq 0^\circ C \) and 1 atm.

Initially, there are mole fractions of ice and liquid water \( f_{\text{init}}^{\text{liq}} \) and \( f_{\text{init}}^{\text{sol}} \) and each at different temperatures \( T_{\text{init}}^{\text{liq}} \) and \( T_{\text{init}}^{\text{sol}} \). They will have equilibrium values \( f_{\text{equil}}^{\text{liq}} \) and \( f_{\text{equil}}^{\text{sol}} \) and a temperature \( T_{\text{equil}} \).

1. Suppose that the initial mole fractions of ice and liquid water are \( f_{\text{init}}^{\text{sol}} = 3/4 \) and \( f_{\text{init}}^{\text{liq}} = 1/4 \) and their initial temperatures are \( T_{\text{init}}^{\text{sol}} = -50^\circ C \) and \( T_{\text{init}}^{\text{liq}} = 100^\circ C \). Compute the equilibrium state.

2. Suppose that the initial mole fractions of ice and liquid water are \( f_{\text{init}}^{\text{sol}} = 3/4 \) and \( f_{\text{init}}^{\text{liq}} = 1/4 \) and their initial temperatures are \( T_{\text{init}}^{\text{sol}} = -5^\circ C \) and \( T_{\text{init}}^{\text{liq}} = 0^\circ C \). Compute the equilibrium state.

3. The challenge in this part of the problem is to visualize the equilibrium states as a function of the input phase fractions and their temperatures.
Characterize and illustrate the equilibrium state of the system as a function of phase fractions of the inserted liquid, $f_l$, and ice, $f_s$.

Your objective is to compute and display an easy-to-interpret graphic (perhaps with input sliders, colors, etc.). The audience (i.e., the reader) should be able to use your graphic and be able to quickly visually determine the equilibrium phase fractions and their temperature as a function of the input data.

**Group Exercise G4-2**

The goal of this problem is to evaluate if a charged particle with charge $q$ of mass $M$ under the influence of a gravitational force $-Mg$ can be “trapped” by an arrangement of “charged patches.”

![Diagram of charged patches and particle](image)

Spherical (negligible radius) particle of mass $M$ under force of gravity with charge $q$ over square charged patches with outer shell of charge density $2\sigma$ (giving repulsive forces) and the inner shell $\sigma$.

1. Determine, if the particle is constrained to lie on the positive $z$-axis, whether the particle can be “stably” levitated?

2. If so, compute the frequency for small amplitude oscillations in the $z$-direction.

3. Determine if the constraint of the $z$-axis is removed (i.e., the particle’s $x$ and $y$ positions can be non-zero), if the particle remains stable.

4. If so, determine the frequencies of small oscillations parallel to the $x$-axis and parallel to the line $x = y$. 
