Find a dimensionless form of Hooke's Law.

I want to make the spring equation \( F = -k(x-x_0) \) non-dimensional, so that I can plot the behavior of any spring regardless of its initial length or its spring constant on a single plot. In order to non-dimensionalize the equation, I need to think about the units of each part of the expression. Both \( x \) and \( x_0 \) have units of length, which, in MKS units, is meters (m). \( F \) has MKS units of Newtons (N). Finally, \( k \) has units Newtons/meter (N/m). The problem statement suggests that I could use \( x_0 \) as the characteristic length and \( kx_0 \) as the characteristic force. Let me use the following notation to represent the relevant quantities:

- \( f \) = non-dimensionalized force
- \( c \) = non-dimensionalized length

It turns out that I don't need to do anything with \( k \) - it will cancel out on its own if I non-dimensionalize the other quantities.

\[
f = \frac{F}{kx_0}
\]

\[
c = \frac{x}{x_0}
\]

\[
c_0 = \frac{x_0}{x_0} = 1
\]

By rearranging the above expressions, I can write replacement rules to use in non-dimensionalizing the original equation.

\[
F \rightarrow f (k x_0)
\]

\[
x \rightarrow c x_0
\]

\[
x_0 \rightarrow c_0 x_0
\]

With the replacement rules determined, let me enter the equation into Mathematica and apply the replacement rules.

1

\[
F = -k (x - x_0)
\]

Now that Hooke's law has been entered, I rewrite the equation using the expression for \( f \) and the replacement rules for \( x \) and \( x_0 \).

2

\[
\phi = \frac{F}{(k x_0)} \cdot \{x \rightarrow \chi x_0, x_0 \rightarrow \chi^0 x_0\}
\]

Notice here that \( k \) is gone. Now, I just need to tell Mathematica that \( \chi_0 \) is 1 and simplify everything.

3

\[
\phi = \phi / \{\chi^0 \rightarrow 1\} // \text{Simplify}
\]

1 - \( \chi \)

The above tells me that the non-dimensionalized version of Hooke's law is \( f = 1 - \chi \) where \( f \) is the force and \( \chi \) is the extension.
Visualize the force of any spring at any length.

Here I just plot the function I found above.

```math
Plot[\phi, \{x, 0, 2\}, PlotLabel \rightarrow 
"Dimensionless Force vs. Dimensionless Displacement for a Hookean Spring",
AxesLabel \rightarrow \{"dimensionless displacement", "dimensionless force"\}]
```

Note that in the above plot, 1 on the x-axis corresponds to x0. Thus, when x < 1, the spring is being compressed. When x > 1, the spring is being extended.

Energy Visualize the stored elastic energy (i.e., potential energy) of any spring at a

From previous work in the class, I know that the derivative of an energy is a force. Specifically, the force is the derivative of the energy times negative one. Thus, since I have the force, in order to find the energy, I need to integrate minus the force.

\[ U = \int -\phi \, dx \]

\[ -\chi + \frac{\chi^2}{2} \]

The problem statement says that the equation for the energy should be constructed such that the energy of the spring at x=x0 is zero. In the non-dimensionalized formulation of the equation, this corresponds to \( \chi = 1 \). So, I need to apply this constraint to the equation above. In the above equation, when \( \chi = 1 \), \( U = -1/2 \). So, I need to add 1/2 to the equation in order to ensure that \( U = 0 \) when \( \chi = 1 \).
Let me check it to make sure it works.

```
U = -\chi + \chi^2 + \frac{1}{2}
```

Let me check it to make sure it works.

```
U /. \chi \to 1
```

Alright, that works. Now, let me plot it.

```math
Plot[U, \{\chi, 0, 2\},
PlotLabel \to "Dimensionless Stored Elastic Energy vs. Dimensionless Displacement for a Hookean Spring", AxesLabel \to
{"dimensionless displacement", "dimensionless stored elastic energy"}]
```

Here we see that we get a parabola. This implies that compression stores as much elastic energy as elongation. The parabolic form of the energy for a spring obeying Hooke's law is the basis for a class of systems known as harmonic oscillators.