_____Nov. 07 2005:   **Lecture 19:** _____

Ordinary Differential Equations: Introduction

Reading:
Kreyszig Sections: §1.1 (pp:2–8) , §1.2 (pp:10–12) , §1.3 (pp:14–18)

_____ **Differential Equations: Introduction** _____

Ordinary differential equations are relations between a function of a single variable, its derivatives, and the variable:

$$F\left(\frac{d^n y(x)}{dx^n}, \frac{d^{n-1} f(x)}{dx^{n-1}}, \ldots, \frac{d^2 y(x)}{dx^2}, \frac{dy(x)}{dx}, y(x), x\right) = 0 \tag{19-1}$$

A *first-order* Ordinary Differential Equation (ODE) has only first derivatives of a function.

$$F(\frac{dy(x)}{dx}, y(x), x) = 0 \tag{19-2}$$

A *second-order* ODE has second and possibly first derivatives.

$$F\left(\frac{d^2 y(x)}{dx^2}, \frac{dy(x)}{dx}, y(x), x\right) = 0 \tag{19-3}$$

For example, the one-dimensional time-independent Shrödinger equation,

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$
$$\text{or}$$
$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) - E\psi(x) = 0$$

is a second-order ordinary differential equation that specifies a relation between the wave function, $\psi(x)$, its derivatives, and a spatially dependent function $U(x)$.

Differential equations result from physical models of anything that varies—whether in space, in time, in value, in cost, in color, etc. For example, differential equations exist for modeling quantities such as: volume, pressure, temperature, density, composition, charge density, magnetization, fracture strength, dislocation density, chemical potential, ionic concentration, refractive index, entropy, stress, etc. That is, almost all models for physical quantities are formulated with a differential equation.

The following example illustrates how some first-order equations arise:

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Iteration: First-Order Sequences
Consider a function that changes according to its current size–that is, at a subsequent iteration, the function grows or shrinks according to how large it is currently.

$$F_{i+1} = F_i + \alpha F_i$$

which is equivalent to

$$F_i = F_{i-1} + \alpha F_{i-1}$$

Iteration Trajectories

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First-Order Finite Differences
The example above is not terribly useful because the change at each increment is an integer and the function only has values for integers. To generalize, a forward difference can be added that allows the variable of the function to "go forward" at an arbitrarily small increment, δ.
If the rate of change of y, is a function $f(y)$ of the current value, then,

$$y_{i+1} = y_i + \delta f(y_i)$$

or

$$y(x = (i+1)\delta)) = y(x = i\delta) + \delta f(y(x = i\delta))$$

where x plays the role of an 'indexed grid' with small separations δ.
Finite Differences

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First-Order Operators

The forward-difference equation considered above relates the next iteration, y_{i+1} to the current value y_i. Only y_i appear on the right-hand-side of the equation—the right-hand-side can be thought of an "Operation" on y that pushes it to the next iteration, i.e.,

$$y_{i+1} = \mathcal{F}(y_i)$$

In this way, the nth iteration is determined from the initial value with

$$y_n = \mathcal{F}^n(y_0)$$

Increment Operators

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_____ **Geometrical Interpretation of Solutions** _____

The relationship between a function and its derivatives for a first-order ODE,

$$F(\frac{dy(x)}{dx}, y(x), x) = 0 \tag{19-4}$$

can be interpreted as a level set formulation for a two-dimensional surface embedded in a three-dimensional space with coordinates (y', y, x). The surface specifies a relationship that must be satisfied between the three coordinates.

If $y'(x)$ can be solved for exactly,

$$\frac{dy(x)}{dx} = f(x, y) \tag{19-5}$$

then $y'(x)$ can be thought of as a height above the x-y plane.

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MATHEMATICA® Example: (notebook) Lecture-19

The Geometry of First-Order ODES: Examples

Consider Newton's law of cooling that states that the rate that a body cools by radiation is proportional to the difference in temperature between the body and its surroundings:

$$\frac{dT(t)}{dt} = -k(T - T_o)$$

Make the equation simpler by converting to a non-dimensional form, let $\Theta = T/T_o$ and $\tau = t/k$, then

$$\frac{d\Theta(\tau)}{d\tau} = (1 - \Theta)$$

Flows

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Separable Equations

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If a first-order ordinary differential equation $F(y', y, x) = 0$ can be rearranged so that only one variable, for instance y, appears on the left-hand-side multiplying its derivative and the other, x, appears only on the right-hand-side, then the equation is said to be 'separated."

$$g(y)\frac{dy}{dx} = f(x) \tag{19-6}$$
$$g(y)dy = f(x)dx$$

Each side of such an equation can be integrated with respect to the variable that appears on that side:

$$\int_{y(x_o)}^{y} g(\eta)d\eta = \int_{x_o}^{x} f(\xi)d\xi \tag{19-7}$$

if the initial value, $y(x_o)$ is known. If not, the equation can be solved with an integration constant C_0,

$$\int g(y)dy = \int f(x)dx + C_0 \tag{19-8}$$

where C_0 is determined from initial conditions.

MATHEMATICA® Example: (notebook) Lecture-19

Using MATHEMATICA© **'s Built-in Ordinary Differential Equation Solver**

MATHEMATICA® has built-in exact and numerical differential equations solvers. DSolve takes a representation of a differential equation with initial and boundary conditions and returns a solution if it can find one. If insufficient initial or boundary conditions are specified, then "integration constants" are added to the solution.

DSolve[]

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While the accuracy of the first-order differencing scheme can be determined by comparison to an exact solution, the question remains of how to establish accuracy and convergence with the step-size δ for an arbitrary ODE. This is a question of primary importance and studied by Numerical Analysis.