

Nov. 02 2005: Lecture 17:

## Function Representation by Fourier Series

Reading:

Kreyszig Sections: §10.1 (pp:527–28), §10.2 (pp:529–36), §10.3 (pp:537–40), §10.4 (pp:541–46)

### Periodic Functions

Periodic functions should be familiar to everyone. The keeping of time, the ebb and flow of tides, the patterns and textures of our buildings, decorations, and vestments invoke repetition and periodicity that seem to be inseparable from the elements of human cognition.<sup>8</sup>

A function that is periodic in a single variable can be expressed as:

$$\begin{aligned} f(x + \lambda) &= f(x) \\ f(t + \tau) &= f(t) \end{aligned} \tag{17-1}$$

The first form is a suggestion of a spatially periodic function with wavelength  $\lambda$  and the second form suggests a function that is periodic in time with period  $\tau$ . Of course, both forms are identical and express that the function has the same value at an infinite number of points ( $x = n\lambda$  in space or  $t = n\tau$  in time where  $n$  is an integer.)

Specification of a periodic function,  $f(x)$ , within one period  $x \in (x_0, x_0 + \lambda)$  defines the function everywhere. The most familiar periodic functions are the trigonometric functions:

$$\sin(x) = \sin(x + 2\pi) \quad \text{and} \quad \cos(x) = \cos(x + 2\pi) \tag{17-2}$$

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#### Making Periodic Functions

Periodic functions are often associated with the “modulus” operation.  $\text{Mod}[x, \lambda]$  is the remainder of dividing  $x$  by  $\lambda$ . Its result always lies in the domain  $0 \leq \text{Mod}[x, \lambda] \leq \lambda$ . Another way to think of modulus is to find the “point” where a periodic function should be evaluated if its primary domain is  $x \in (0, \lambda)$ .

*Periodic extensions of functions*

<sup>8</sup>I hope you enjoy the lyrical quality of the prose. While I wonder again if anyone is reading these notes, my wistfulness is taking a poetic turn:

They repeat themselves  
 What is here, will be there  
 It wills, willing, to be again  
 spring; neap, ebb and flow, wane; wax  
 sow; reap, warp and woof, motif; melody.  
 The changed changes. We remain  
 Perpetually, Immutably, Endlessly.

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**Odd and Even Functions**

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The trigonometric functions have the additional properties of being an *odd* function about the point  $x = 0$ :  $f_{\text{odd}} : f_{\text{odd}}(x) = -f_{\text{odd}}(-x)$  in the case of the sine, and an *even* function in the case of the cosine:  $f_{\text{even}} : f_{\text{even}}(x) = f_{\text{even}}(-x)$ .

This can be generalized to say that a function is even or odd about a point  $\lambda/2$ :  $f_{\text{odd}}^{\lambda/2} : f_{\text{odd}}^{\lambda/2}(\lambda/2 + x) = -f_{\text{odd}}^{\lambda/2}(\lambda/2 - x)$  and  $f_{\text{even}}^{\lambda/2} : f_{\text{even}}^{\lambda/2}(\lambda/2 + x) = f_{\text{even}}^{\lambda/2}(\lambda/2 - x)$ .

Any function can be decomposed into an odd and even sum:

$$g(x) = g_{\text{even}} + g_{\text{odd}} \quad (17-3)$$


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The sine and cosine functions can be considered the odd and even parts of the generalized trigonometric function:

$$e^{ix} = \cos(x) + i \sin(x) \quad (17-4)$$

with period  $2\pi$ .

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**Representing a particular function with a sum of other functions**

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A Taylor expansion approximates the behavior of a suitably defined function,  $f(x)$  in the neighborhood of a point,  $x_o$ , with a bunch of functions,  $p_i(x)$ , defined by the set of powers:

$$p_i \equiv \vec{p} = (x^0, x^1, \dots, x^j, \dots) \quad (17-5)$$

The polynomial that approximates the function is given by:

$$f(x) = \vec{A} \cdot \vec{p} \quad (17-6)$$

where the vector of coefficients is defined by:

$$A_i \equiv \vec{A} = \left( \frac{1}{0!} f(x_o), \frac{1}{1!} \left. \frac{df}{dx} \right|_{x_o}, \dots, \frac{1}{j!} \left. \frac{d^j f}{dx^j} \right|_{x_o}, \dots \right) \quad (17-7)$$

The idea of a vector of infinite length has not been formally introduced, but the idea that as the number of terms in the sum in Eq. 17-6 gets larger and larger, the approximation should converge to the function. In the limit of an infinite number of terms in the sum (or the vectors of infinite length) the series expansion will converge to  $f(x)$  if it satisfies some technical continuity constraints.

However, for periodic functions, the domain over which the approximation is required is only one period of the periodic function—the rest of the function is taken care of by the definition of periodicity in the function.

Because the function is periodic, it makes sense to use functions that have the same period to approximate it. The simplest periodic functions are the trigonometric functions. If the period is  $\lambda$ , any other periodic function with periods  $\lambda/2, \lambda/3, \lambda/N$ , will also have period  $\lambda$ . Using these "sub-periodic" trigonometric functions is the idea behind Fourier Series.

## Fourier Series

The functions  $\cos(2\pi x/\lambda)$  and  $\sin(2\pi x/\lambda)$  each have period  $\lambda$ . That is, they each take on the same value at  $x$  and  $x + \lambda$ .

There are an infinite number of other simple trigonometric functions that are periodic in  $\lambda$ ; they are  $\cos[2\pi x/(\lambda/2)]$  and  $\sin[2\pi x/(\lambda/2)]$  and which cycle two times within each  $\lambda$ ,  $\cos[2\pi x/(\lambda/3)]$  and  $\sin[2\pi x/(\lambda/3)]$  and which cycle three times within each  $\lambda$ , and, in general,  $\cos[2\pi x/(\lambda/n)]$  and  $\sin[2\pi x/(\lambda/n)]$  and which cycle  $n$  times within each  $\lambda$ .

The constant function,  $a_0(x) = \text{const.}$ , also satisfies the periodicity requirement.

The superposition of multiples of any number of periodic function must also be a periodic function, therefore any function  $f(x)$  that satisfies:

$$f(x) = \mathcal{E}_0 + \sum_{n=1}^{\infty} \mathcal{E}_n \cos\left(\frac{2\pi n}{\lambda}x\right) + \sum_{n=1}^{\infty} \mathcal{O}_n \sin\left(\frac{2\pi n}{\lambda}x\right) \\ = \mathcal{E}_{k_0} + \sum_{n=1}^{\infty} \mathcal{E}_{k_n} \cos(k_n x) + \sum_{n=1}^{\infty} \mathcal{O}_{k_n} \sin(k_n x) \quad (17-8)$$

where the  $k_i$  are the *wave-numbers* or *reciprocal wavelengths* defined by  $k_j \equiv 2\pi j/\lambda$ . The  $k$ 's represent inverse wavelengths—large values of  $k$  represent short-period or high-frequency terms.

If any periodic function  $f(x)$  could be represented by the series in Eq. 17-8 by a suitable choice of coefficients, then an alternative representation of the periodic function could be obtained in terms of the simple trigonometric functions and their amplitudes.

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The “inverse question” remains: “How are the amplitudes  $\mathcal{E}_{k_n}$  (the even trigonometric terms) and  $\mathcal{O}_{k_n}$  (the odd trigonometric terms) determined for a given  $f(x)$ ?”

The method follows from what appears to be a “trick.” The following three integrals have simple forms for integers  $M$  and  $N$ :

$$\begin{aligned} \int_{x_0}^{x_0+\lambda} \sin\left(\frac{2\pi M}{\lambda}x\right) \sin\left(\frac{2\pi N}{\lambda}x\right) dx &= \begin{cases} \frac{\lambda}{2} & \text{if } M = N \\ 0 & \text{if } M \neq N \end{cases} \\ \int_{x_0}^{x_0+\lambda} \cos\left(\frac{2\pi M}{\lambda}x\right) \cos\left(\frac{2\pi N}{\lambda}x\right) dx &= \begin{cases} \frac{\lambda}{2} & \text{if } M = N \\ 0 & \text{if } M \neq N \end{cases} \\ \int_{x_0}^{x_0+\lambda} \cos\left(\frac{2\pi M}{\lambda}x\right) \sin\left(\frac{2\pi N}{\lambda}x\right) dx &= 0 \text{ for any integers } M, N \end{aligned} \quad (17-9)$$

Therefore, any amplitude can be determined by multiplying both sides of Eq. 17-8 by its conjugate trigonometric function and integrating over the domain. (Here we pick the domain to start at zero,  $x \in (0, \lambda)$ , but any other starting point would work fine.)

$$\begin{aligned} \cos(k_M x) f(x) &= \cos(k_M x) \left( \mathcal{E}_{k_0} + \sum_{n=1}^{\infty} \mathcal{E}_{k_n} \cos(k_n x) + \sum_{n=1}^{\infty} \mathcal{O}_{k_n} \sin(k_n x) \right) \\ \int_0^{\lambda} \cos(k_M x) f(x) dx &= \int_0^{\lambda} \cos(k_M x) \left( \mathcal{E}_{k_0} + \sum_{n=1}^{\infty} \mathcal{E}_{k_n} \cos(k_n x) + \sum_{n=1}^{\infty} \mathcal{O}_{k_n} \sin(k_n x) \right) dx \quad (17-10) \\ \int_0^{\lambda} \cos(k_M x) f(x) dx &= \frac{\lambda}{2} \mathcal{E}_{k_M} \end{aligned}$$

This provides a formula to calculate the even coefficients (amplitudes) and multiplying by a sin function provides a way to calculate the odd coefficients (amplitudes) for  $f(x)$  periodic in the fundamental domain  $x \in (0, \lambda)$ .

$$\begin{aligned} \mathcal{E}_{k_0} &= \frac{1}{\lambda} \int_0^{\lambda} f(x) dx \\ \mathcal{E}_{k_N} &= \frac{2}{\lambda} \int_0^{\lambda} f(x) \cos(k_N x) dx \quad k_N \equiv \frac{2\pi N}{\lambda} \\ \mathcal{O}_{k_N} &= \frac{2}{\lambda} \int_0^{\lambda} f(x) \sin(k_N x) dx \quad k_N \equiv \frac{2\pi N}{\lambda} \end{aligned} \quad (17-11)$$

The constant term has an extra factor of two because  $\int_0^\lambda \mathcal{E}_{k_0} dx = \lambda \mathcal{E}_{k_0}$  instead of the  $\lambda/2$  found in Eq. 17-9.

## MATHEMATICA® Example: (notebook) Lecture-17

**Orthogonality of Trigonometric Functions**

Demonstrate that the relations in Eq. 17-9 are true.

MATHEMATICA® does not directly simplify and cancel periodic sine and cosine terms.

**Calculating Fourier Amplitudes**

1. Write functions to calculate the even (cosine) coefficients and the odd (sine) coefficients using the formulas in Eq. 17-11.
2. It is convenient to include the zeroth-order coefficient for the odd (sine) series which vanishes by definition.
3. The functions work by doing an integral for each term—this is not very efficient. It would be more efficient to calculate the integral symbolically once and then evaluate it once for each term.
4. Define an example function,  $f(x) = x(1-x)^2(2-x)$ , with which to demonstrate the Fourier approximation. This function is even about the  $x = 1$  point.
5. Define functions that create vectors of amplitudes for the cosine and sine terms.
6. If a wavelength  $\lambda = 2$  is used for the domain, then we should see that all odd (sine) Fourier amplitudes should vanish.
7. The Fourier series up to a certain order can be defined as the sum of two inner (dot) products: the inner product of the odd coefficient vector and the sine basis vector, and the inner product of the even coefficient vector and the cosine basis vector.
8. Illustrate convergence for different orders of approximation.

**Using the Fourier Package**

Of course, Fourier series expansions are a common and useful mathematical tool, and it is not surprising that MATHEMATICA® would have a package to do this and replace the inefficient functions defined above.

1. MATHEMATICA® 's Fourier package is designed to operate on the unit period located at  $x \in (-1/2, 1/2)$ .
2. Construct a function that converts a function defined as periodic in fundamental domain  $x \in (0, \lambda)$  and maps it to a function that has fundamental domain  $x \in (-1/2, 1/2)$ .
3. Demonstrate how MATHEMATICA® 's Fourier functions operate and illustrate convergence.
4. If the function being approximated has some discontinuities either in its values, or in its first derivatives, then wiggles will develop near the discontinuities where the convergence is not locally well-behaved. This is called Gibbs' phenomenon.
5. Use the Fourier package to illustrate and interpret the the spectrum of Fourier coefficients.

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### Other forms of the Fourier coefficients

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Sometimes the primary domain is defined with a different starting point and different symbols, for instance Kreyszig uses a centered domain by using  $-L$  as the starting point and  $2L$  as the period, and in these cases the forms for the Fourier coefficients look a bit different. One needs to look at the domain in order to determine which form of the formulas to use.

#### *Extra Information and Notes*

##### **Potentially interesting but currently unnecessary**

*The “trick” of multiplying both sides of Eq. 17-8 by a function and integrating comes from the fact that the trigonometric functions form an orthogonal basis for functions with inner product defined by*

$$f(x) \cdot g(x) = \int_0^\lambda f(x)g(x)dx$$

*Considering the trigonometric functions as components of a vector:*

$$\begin{aligned} \vec{e}_0(x) &= (1, 0, 0, \dots, ) \\ \vec{e}_1(x) &= (0, \cos(k_1 x), 0, \dots, ) \\ \vec{e}_2(x) &= (0, 0, \sin(k_1 x), \dots, ) \\ &\dots = \quad \vdots \\ \vec{e}_n(x) &= (\dots, \sin(k_n x), \dots, ) \end{aligned}$$

*then these “basis vectors” satisfy  $\vec{e}_i \cdot \vec{e}_j = (\lambda/2)\delta_{ij}$ , where  $\delta_{ij} = 0$  unless  $i = j$ . The trick is just that, for an arbitrary function represented by the basis vectors,  $\vec{P}(x) \cdot \vec{e}_j(x) = (\lambda/2)P_j$ .*

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### Complex Form of the Fourier Series

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The behavior of the Fourier coefficients for both the odd (sine) and for the even (cosine) terms was illustrated above. Functions that are even about the center of the fundamental domain (reflection symmetry) will have only even terms—all the sine terms will vanish. Functions that are odd about the center of the fundamental domain (reflections across the center of the domain and then across the  $x$ -axis.) will have only odd terms—all the cosine terms will vanish.

Functions with no odd or even symmetry will have both types of terms (odd and even) in its expansion. This is a restatement of the fact that any function can be decomposed into odd and even parts (see Eq. 17-3).

This suggests a short-hand in Eq. 17-4 can be used that combines both odd and even series into one single form. However, because the odd terms will all be multiplied by the imaginary number  $i$ , the coefficients will generally be complex. Also because  $\cos(nx) = (\exp(inx) + \exp(-inx))/2$ , writing the sum in terms of exponential functions only will require that the sum must be over both positive and negative integers.

For a periodic domain  $x \in (0, \lambda)$ ,  $f(x) = f(x + \lambda)$ , the *complex form of the fourier series*

*is given by:*

$$f(x) = \sum_{n=-\infty}^{\infty} \mathcal{C}_{k_n} e^{\imath k_n x} \quad \text{where } k_n \equiv \frac{2\pi n}{\lambda} \quad (17-12)$$

$$\mathcal{C}_{k_n} = \frac{1}{\lambda} \int_0^\lambda f(x) e^{-\imath k_n x} dx$$

Because of the orthogonality of the basis functions  $\exp(ik_n x)$ , the domain can be moved to any wavelength, the following is also true:

$$f(x) = \sum_{n=-\infty}^{\infty} \mathcal{C}_{k_n} e^{ik_n x} \quad \text{where } k_n \equiv \frac{2\pi n}{\lambda} \quad (17-13)$$

$$\mathcal{C}_{k_n} = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x) e^{-ik_n x} dx$$

although the coefficients may have a different form.