
Oct. 19 2005: **Lecture 14:**

Integrals along a Path

Reading:

Kreyszig Sections: §9.1 (pp:464–70) , §9.2 (pp:471–477) §9.3 (pp:478–484)

Integrals along a Curve

Consider the type of integral that everyone learns initially:

$$E(b) - E(a) = \int_a^b f(x) dx \quad (14-1)$$

The equation implies that f is integrable and

$$dE = f dx = \frac{dE}{dx} dx \quad (14-2)$$

so that the integral can be written in the following way:

$$E(b) - E(a) = \int_a^b dE \quad (14-3)$$

where a and b represent “points” on some *line* where E is to be evaluated.

Of course, there is no reason to restrict integration to a straight line—the generalization is the integration along a curve (or a path) $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$.

$$E(b) - E(a) = \int_{\vec{x}(a)}^{\vec{x}(b)} \vec{f}(\vec{x}) \cdot d\vec{x} = \int_a^b g(x(\vec{t})) dt = \int_a^b \nabla E \cdot \frac{d\vec{x}}{dt} dt = \int_a^b dE \quad (14-4)$$

This last set of equations assumes that the gradient exists—i.e., there is some function E that has the gradient $\nabla E = \vec{f}$.

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Path Dependence, Curl, and Curl=0 subspaces

This example will show that the choice of path matters for a vector-valued function that does not have vanishing curl and that it doesn't matter when integrating a function with vanishing curl.

Path-dependent/Non-conserving Field 1. Verify that the function $\vec{v}(\vec{x}) = xyz(\hat{i} + \hat{k} + \hat{z})$ does not have vanishing curl.

2. Integrate $\vec{v}$ along a path that is wrapped around a cylinder of radius R , (e.g., $(x(t), y(t), z(t)) = (R \cos t, R \sin t, AP_{2\pi}(t))$, where $P_{2\pi}(t = 0) = P_{2\pi}(t = 2\pi)$)

3. Calculate the integral specifically for $P_{2\pi}(t) = \cos t$, $P_{2\pi}(t) = \sin t$, $P_{2\pi}(t) = t(t - 2\pi)$, and $P_{2\pi}(t) = \cos Nt$.

Path-independent/Conservative Field 1. Verify that, for the function $\vec{w}(\vec{x}) = e^{xyz}(yz\hat{i} + zx\hat{k} + xy\hat{z})$, $\nabla \times \vec{w} = 0$. In fact, $\vec{w} = \nabla e^{xyz}$.

2. Integrate $\vec{w}$ along the same cylindrical-type path as above and see that the integral always vanishes—it is path-independent.

Path independent on a Subspace 1. The vector function $\vec{v}(\vec{x}) = (x^2 + y^2 - R^2)\hat{z}$ only vanishes on the cylinder or radius R .

2. It is easy to find $\vec{w}$ such that $\vec{w} = \nabla \times v$:

$$\vec{w} = \frac{1}{2} \left(yR^2 \left[1 - x^2 - \frac{y^2}{3} \right] \hat{x} + -xR^2 \left[1 - y^2 - \frac{x^2}{3} \right] \hat{y} \right)$$

In fact, because we could add any vector function that has vanishing curl to $\vec{w}$ there are an infinite number of $\vec{w}$ such that $\vec{w} = \nabla \times v$.

3. Therefore, if we integrate $\vec{w}$ along a path *that is restricted* to the cylinder it should be path independent.

4. Using the same methods as above, we find that the integral on the cylinder will be independent of P —the vector function $\vec{w}$ is independent of path as long as the path remains on the cylinder.

Multidimensional Integrals

Perhaps the most straightforward of the higher-dimensional integrations (e.g., vector function along a curve, vector function on a surface) is a scalar function over a domain such as, a rectangular block in two dimensions, or a block in three dimensions. In each case, the integration over a dimension is uncoupled from the others and the problem reduces to pedestrian integration along a coordinate axis.

Sometimes difficulty arises when the domain of integration is not so easily described; in these cases, the limits of integration become functions of another integration variable. While specifying the limits of integration requires a bit of attention, the only thing that makes these cases difficult is that the integrals become tedious and lengthy. MATHEMATICA[®] removes

some of this burden.

A short review of various ways in which a function's variable can appear in an integral follows:

	The Integral	Its Derivative
Function of limits	$p(x) = \int_{\alpha(x)}^{\beta(x)} f(\xi) d\xi$	$\frac{dp}{dx} = f(\beta(x)) \frac{d\beta}{dx} - f(\alpha(x)) \frac{d\alpha}{dx}$
Function of integrand	$q(x) = \int_a^b g(\xi, x) d\xi$	$\frac{dq}{dx} = \int_a^b \frac{\partial g(\xi, x)}{\partial x} d\xi$
Function of both	$r(x) = \int_{\alpha(x)}^{\beta(x)} g(\xi, x) d\xi$	$\begin{aligned} \frac{dr}{dx} = & f(\beta(x)) \frac{d\beta}{dx} - f(\alpha(x)) \frac{d\alpha}{dx} \\ & + \int_{\alpha(x)}^{\beta(x)} \frac{\partial g(\xi, x)}{\partial x} d\xi \end{aligned}$

Extra Information and Notes

Potentially interesting but currently unnecessary

Changing of variables is a topic in multivariable calculus that often causes difficulty in classical thermodynamics.

This is an extract of my notes on thermodynamics: <http://pruffle.mit.edu/3.00/>

Alternative forms of differential relations can be derived by changing variables.

To change variables, a useful scheme using Jacobians can be employed:

$$\begin{aligned}
 \frac{\partial(u, v)}{\partial(x, y)} &\equiv \det \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
 &= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \\
 &= \left(\frac{\partial u}{\partial x} \right)_y \left(\frac{\partial v}{\partial y} \right)_x - \left(\frac{\partial u}{\partial y} \right)_x \left(\frac{\partial v}{\partial x} \right)_y \\
 &= \frac{\partial u(x, y)}{\partial x} \frac{\partial v(x, y)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \frac{\partial v(x, y)}{\partial x}
 \end{aligned} \tag{14-9}$$

$$\begin{aligned}
 \frac{\partial(u, v)}{\partial(x, y)} &= -\frac{\partial(v, u)}{\partial(x, y)} = \frac{\partial(v, u)}{\partial(y, x)} \\
 \frac{\partial(u, v)}{\partial(x, v)} &= \left(\frac{\partial u}{\partial x} \right)_v \\
 \frac{\partial(u, v)}{\partial(x, y)} &= \frac{\partial(u, v)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(x, y)}
 \end{aligned} \tag{14-10}$$

For example, the heat capacity at constant volume is:

$$\begin{aligned}
 C_V &= T \left(\frac{\partial S}{\partial T} \right)_V = T \frac{\partial(S, V)}{\partial(T, V)} \\
 &= T \frac{\partial(S, V)}{\partial(T, P)} \frac{\partial(T, P)}{\partial(T, V)} = T \left[\left(\frac{\partial S}{\partial T} \right)_P \left(\frac{\partial V}{\partial P} \right)_T - \left(\frac{\partial S}{\partial P} \right)_T \left(\frac{\partial V}{\partial T} \right)_P \right] \left(\frac{\partial P}{\partial V} \right)_T \\
 &= T \frac{C_P}{T} - T \left(\frac{\partial P}{\partial V} \right)_T \left(\frac{\partial V}{\partial T} \right)_P \left(\frac{\partial S}{\partial P} \right)_T
 \end{aligned} \tag{14-11}$$

Using the Maxwell relation, $\left(\frac{\partial S}{\partial P} \right)_T = - \left(\frac{\partial V}{\partial T} \right)_P$,

$$C_P - C_V = -T \frac{\left[\left(\frac{\partial V}{\partial T} \right)_P \right]^2}{\left(\frac{\partial V}{\partial P} \right)_T} \tag{14-12}$$

which demonstrates that $C_P > C_V$ because, for any stable substance, the volume is a decreasing function of pressure at constant temperature.

MATHEMATICA[®] Example: (notebook) Lecture-14

Potential near a Charged and Shaped Surface Patch

Example calculation of the spatially-dependent energy of a unit point charge in the vicinity of a charged planar region having the shape of an equilateral triangle.

The energy of a point charge $|e|$ due to a surface patch on the plane $z = 0$ of size $d\xi d\eta$ with surface charge density $\sigma(x, y)$ is:

$$dE(x, y, z, \xi, \eta) = \frac{|e|\sigma(\xi, \eta)d\xi d\eta}{\vec{r}(x, y, z, \xi, \eta)}$$

for a patch with uniform charge,

$$dE(x, y, z, \xi, \eta) = \frac{|e|\sigma d\xi d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}}$$

For an equilateral triangle with sides of length one and center at the origin, the vertices can be located at $(0, \sqrt{3}/2)$ and $(\pm 1/2, -\sqrt{3}/6)$.

The integration becomes

$$E(x, y, z) \propto \int_{-\sqrt{3}/6}^{\sqrt{3}/2} \left(\int_{\eta-\sqrt{3}/2}^{\sqrt{3}/2-\eta} \frac{d\xi}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}} \right) d\eta$$

MATHEMATICA[®] 's syntax is to integrate over the last integration iterator first, and the first iterator last; i.e., the expression:

`Integrate[1/r[x,y,z], {x,a,b}, {y,f[x],g[x]}, {z,p[x,y],q[x,y]}]`
would integrate over z first, y second, and lastly x .

The closed form of the above integral appears to be unknown to MATHEMATICA[®]. However, the energy can be integrated numerically without difficulty and visualized.