
 Oct. 17 2005: Lecture 13:

Differential Operations on Vectors

Reading:

Kreyszig Sections: §8.10 (pp:453–56) , §8.11 (pp:457–459)

Generalizing the Derivative

The number of different ideas, whether from physical science or other disciplines, that can be understood with reference to the “meaning” of a derivative from the calculus of scalar functions is very very large. Our ideas about many topics, such as price elasticity, strain, stability, and optimization, are connected to our understanding of a derivative.

In vector calculus, there are generalizations to the derivative from basic calculus that acts on a scalar and gives another scalar back:

gradient (∇): A derivative on a scalar that gives a vector.

curl ($\nabla \times$): A derivative on a vector that gives another vector.

divergence ($\nabla \cdot$): A derivative on a vector that gives scalar.

Each of these have “meanings” that can be applied to a broad class of problems.

The gradient operation on $f(\vec{x}) = f(x, y, z) = f(x_1, x_2, x_3)$,

$$\text{grad}f = \nabla f \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f \quad (13-1)$$

has been discussed previously. The curl and divergence will be discussed below.

MATHEMATICA® Example: (notebook) Lecture-13
Gradient of a several $1/r$ potentials

Three Electric Charges

Coordinate Systems

The above definitions are for a Cartesian (x, y, z) system. Sometimes it is more convenient to

work in other (spherical, cylindrical, etc) coordinate systems. In other coordinate systems, the derivative operations ∇ , $\nabla \cdot$, and $\nabla \times$ have different forms. These other forms can be derived, or looked up in a mathematical handbook, or specified by using the MATHEMATICA® package “VectorAnalysis.”

MATHEMATICA® Example: (notebook) Lecture-13

Coordinate System Transformations

Converting between Cartesian and Spherical Coordinates with MATHEMATICA®

The divergence operates on a vector field that is a function of position, $\vec{v}(x, y, z) = \vec{v}(\vec{x}) = (v_1(\vec{x}), v_2(\vec{x}), v_3(\vec{x}))$, and returns a scalar that is a function of position. The scalar field is often called the divergence field of \vec{v} or simply the divergence of \vec{v} .

$$\text{div } \vec{v}(\vec{x}) = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (v_1, v_2, v_3) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \vec{v} \quad (13-2)$$

Think about what the divergence means,

Curl and Its Interpretation

The curl is the vector valued derivative of a vector function. As illustrated below, its operation can be geometrically interpreted as the rotation of a field about a point.

For a vector-valued function of (x, y, z) :

$$\vec{v}(x, y, z) = \vec{v}(\vec{x}) = (v_1(\vec{x}), v_2(\vec{x}), v_3(\vec{x})) = v_1(x, y, z)\hat{i} + v_2(x, y, z)\hat{j} + v_3(x, y, z)\hat{k} \quad (13-3)$$

the curl derivative operation is another vector defined by:

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \left(\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right), \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right), \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \right) \quad (13-4)$$

or with the memory-device:

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{pmatrix} \quad (13-5)$$

MATHEMATICA® Example: (notebook) Lecture-13

Calculating the Curl of a Function

Consider the vector function that is often used in Brakke's Surface Evolver program:

$$\vec{w} = \frac{z^n}{(x^2 + y^2)(x^2 + y^2 + z^2)^{\frac{n}{2}}} (y\hat{i} - x\hat{j})$$

This can be shown easily, using MATHEMATICA®, to have the property:

$$\nabla \times \vec{w} = \frac{nz^{n-1}}{(x^2 + y^2 + z^2)^{1+\frac{n}{2}}} (x\hat{i} + y\hat{j} + z\hat{k})$$

which is spherically symmetric for $n = 1$ and convenient for turning surface integrals over a portion of a sphere into a path-integral over a curve on a sphere.

1. Create vector function \vec{w} above and visualize using the PlotVectorField3D function in MATHEMATICA®'s PlotField3D package.
2. The function will be singular for $n > 1$ along the $z - axis$, this singularity will be communicated during the numerical evaluations for visualization unless some care is applied.
3. Demonstrate the above assertion about \vec{w} and its curl.
4. Visualize the curl: note that the field is points up with large magnitude near the vortex at the origin.
5. Demonstrate that the divergence of the curl of \vec{w} vanishes for any n .

One important result that has physical implications is that the curl of a gradient is always zero: $f(\vec{x}) = f(x, y, z)$:

$$\nabla \times (\nabla f) = 0 \quad (13-6)$$

Therefore if some vector function $\vec{F}(x, y, z) = (F_x, F_y, F_z)$ can be derived from a scalar potential, $\nabla f = \vec{F}$, then the curl of \vec{F} must be zero. This is the property of an exact differential

$df = (\nabla f) \cdot (dx, dy, dz) = \vec{F} \cdot (dx, dy, dz)$. Maxwell's relations follow from equation 13-6:

$$\begin{aligned} 0 &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial \frac{\partial f}{\partial z}}{\partial y} - \frac{\partial \frac{\partial f}{\partial y}}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \\ 0 &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial \frac{\partial f}{\partial x}}{\partial z} - \frac{\partial \frac{\partial f}{\partial z}}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \\ 0 &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial \frac{\partial f}{\partial y}}{\partial x} - \frac{\partial \frac{\partial f}{\partial x}}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \end{aligned} \quad (13-7)$$

Another interpretation is that gradient fields are *curl free, irrotational, or conservative*.

The notion of conservative means that, if a vector function can be derived as the gradient of a scalar potential, then integrals of the vector function over any path is zero for a closed curve—meaning that there is no change in “state;” energy is a common state function.

Here is a picture that helps visualize why the curl invokes names associated with spinning, rotation, etc.

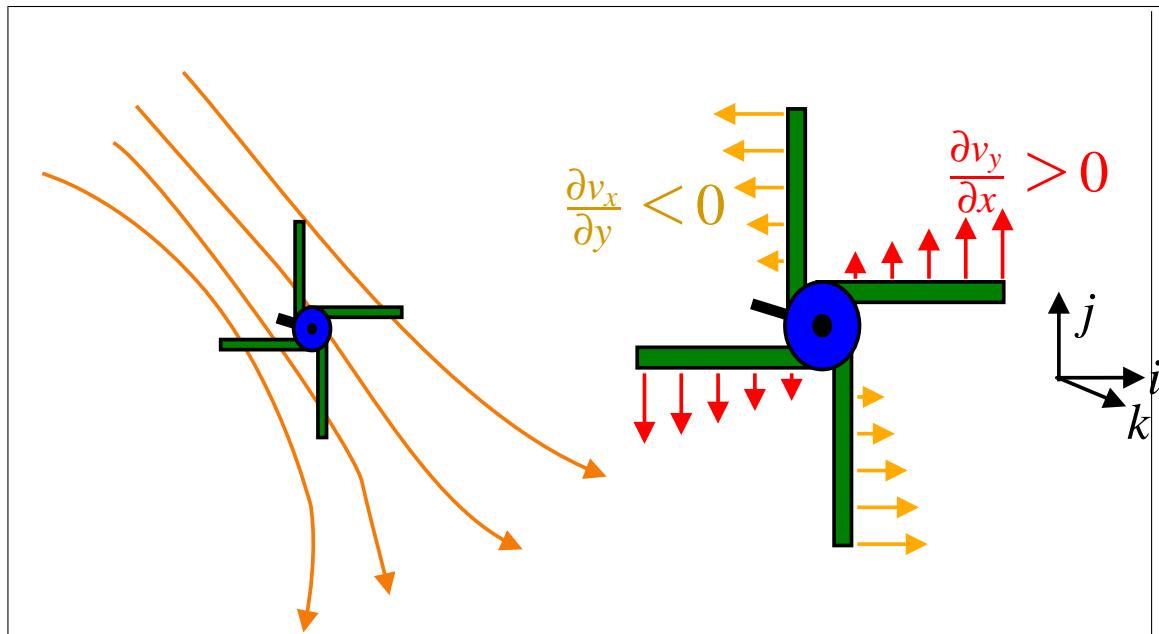


Figure 13-1: Consider a small paddle wheel placed in a set of stream lines defined by a vector field of position. If the v_y component is an increasing function of x , this tends to make the paddle wheel want to spin (positive, counter-clockwise) about the \hat{k} -axis. If the v_x component is a decreasing function of y , this tends to make the paddle wheel want to spin (positive, counter-clockwise) about the \hat{k} -axis. The net impulse to spin around the \hat{k} -axis is the sum of the two.

Note that this is independent of the reference frame because a constant velocity $\vec{v} = \text{const.}$ and the local acceleration $\vec{v} = \nabla f$ can be subtracted because of Eq. 13-8.

Another important result is that divergence of any curl is also zero, for $\nabla \cdot (\nabla \times \vec{v}) = \vec{v}(x, y, z)$:

$$\nabla \cdot (\nabla \times \vec{v}) = 0 \quad (13-8)$$