Oct. 17 2005: Lecture 13:

Differential Operations on Vectors

Reading:
Kreyszig Sections: §8.10 (pp:453–56) , §8.11 (pp:457–459)

Generalizing the Derivative

The number of different ideas, whether from physical science or other disciplines, that can be understood with reference to the "meaning" of a derivative from the calculus of scalar functions is very very large. Our ideas about many topics, such as price elasticity, strain, stability, and optimization, are connected to our understanding of a derivative.

In vector calculus, there are generalizations to the derivative from basic calculus that acts on a scalar and gives another scalar back:

gradient (∇): A derivative on a scalar that gives a vector.

curl ($\nabla\times$): A derivative on a vector that gives another vector.

divergence ($\nabla\cdot$): A derivative on a vector that gives scalar.

Each of these have "meanings" that can be applied to a broad class of problems.

The gradient operation on $f(\vec{x}) = f(x, y, z) = f(x_1, x_2, x_3)$,

$$\text{grad} f = \nabla f \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f \tag{13-1}$$

has been discussed previously. The curl and divergence will be discussed below.

Mathematica® Example: (notebook) Lecture-13
Gradient of a several $1/r$ potentials   *Three Electric Charges*

Divergence and Its Interpretation

Coordinate Systems ....

The above definitions are for a Cartesian (x, y, z) system. Sometimes it is more convenient to work in other (spherical, cylindrical, etc) coordinate systems. In other coordinate systems, the derivative operations ∇, $\nabla\cdot$, and $\nabla\times$ have different forms. These other forms can be derived, or looked up in a mathematical handbook, or specified by using the MATHEMATICA® package "VectorAnalysis."

MATHEMATICA® Example: (notebook) Lecture-13
Coordinate System Transformations
Converting between Cartesian and Spherical Coordinates with MATHEMATICA®

The divergence operates on a vector field that is a function of position, $\vec{v}(x,y,z) = \vec{v}(\vec{x}) = (v_1(\vec{x}), v_2(\vec{x}), v_3(\vec{x}))$, and returns a scalar that is a function of position. The scalar field is often called the divergence field of $\vec{v}$ or simply the divergence of $\vec{v}$.

$$\text{div } \vec{v}(\vec{x}) = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (v_1, v_2, v_3) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \vec{v} \quad (13\text{-}2)$$

Think about what the divergence means,

Curl and Its Interpretation

The curl is the vector valued derivative of a vector function. As illustrated below, its operation can be geometrically interpreted as the rotation of a field about a point.

For a vector-valued function of (x, y, z):

$$\vec{v}(x,y,z) = \vec{v}(\vec{x}) = (v_1(\vec{x}), v_2(\vec{x}), v_3(\vec{x})) = v_1(x,y,z)\hat{\imath} + v_2(x,y,z)\hat{\jmath} + v_3(x,y,z)\hat{k} \quad (13\text{-}3)$$

the curl derivative operation is another vector defined by:

$$\text{curl}\,\vec{v} = \nabla \times \vec{v} = \left(\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right), \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right), \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \right) \tag{13-4}$$

or with the memory-device:

$$\text{curl}\,\vec{v} = \nabla \times \vec{v} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{pmatrix} \tag{13-5}$$

MATHEMATICA® Example: (notebook) Lecture-13

Calculating the Curl of a Function

Consider the vector function that is often used in Brakke's Surface Evolver program:

$$\vec{w} = \frac{z^n}{(x^2+y^2)(x^2+y^2+z^2)^{\frac{n}{2}}}(y\hat{i} - x\hat{j})$$

This can be shown easily, using MATHEMATICA® , to have the property:

$$\nabla \times \vec{w} = \frac{nz^{n-1}}{(x^2+y^2+z^2)^{1+\frac{n}{2}}}(x\hat{i} + y\hat{j} + z\hat{k})$$

which is spherically symmetric for $n = 1$ and convenient for turning surface integrals over a portion of a sphere into a path-integral over a curve on a sphere.

1. Create vector function $\vec{w}$ above and visualize using the PlotVectorField3D function in MATHEMATICA® 's PlotField3D package.
2. The function will be singular for $n > 1$ along the $z - axis$, this singularity will be communicated during the numerical evaluations for visualization unless some care is applied.
3. Demonstrate the above assertion about $\vec{w}$ and its curl.
4. Visualize the curl: note that the field is points up with large magnitude near the vortex at the origin.
5. Demonstrate that the divergence of the curl of $\vec{w}$ vanishes for any n.

One important result that has physical implications is that a the curl of a gradient is always zero: $f(\vec{x}) = f(x, y, z)$:

$$\nabla \times (\nabla f) = 0 \tag{13-6}$$

Therefore *if some vector function* $\vec{F}(x, y, z) = (F_x, F_y, F_z)$ *can be derived from a scalar potential,* $\nabla f = \vec{F}$*, then the curl of* $\vec{F}$ *must be zero.* This is the property of an *exact* differential

$df = (\nabla f) \cdot (dx, dy, dz) = \vec{F} \cdot (dx, dy, dz)$. Maxwell's relations follow from equation 13-6:

$$\begin{aligned}
0 =& \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial \frac{\partial f}{\partial z}}{\partial y} - \frac{\partial \frac{\partial f}{\partial y}}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \\
0 =& \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial \frac{\partial f}{\partial x}}{\partial z} - \frac{\partial \frac{\partial f}{\partial z}}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \\
0 =& \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial \frac{\partial f}{\partial y}}{\partial x} - \frac{\partial \frac{\partial f}{\partial x}}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}
\end{aligned} \tag{13-7}$$

Another interpretation is that gradient fields are *curl free, irrotational, or conservative.*

The notion of conservative means that, if a vector function can be derived as the gradient of a scalar potential, then integrals of the vector function over any path is zero for a closed curve—meaning that there is no change in "state;" energy is a common state function.

Here is a picture that helps visualize why the curl invokes names associated with spinning, rotation, etc.

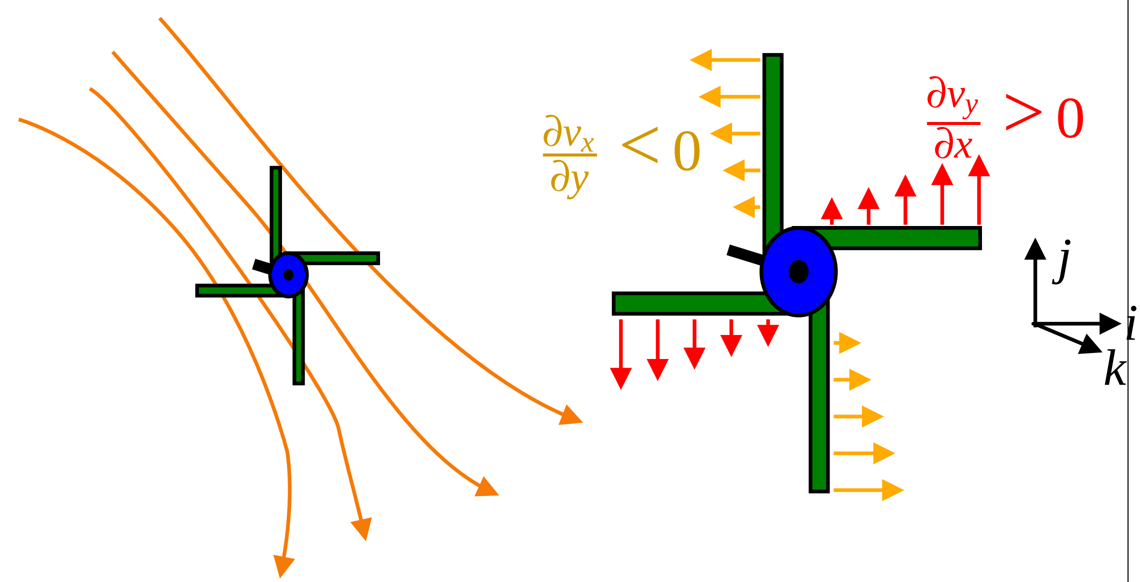

Figure 13-1: Consider a small paddle wheel placed in a set of stream lines defined by a vector field of position. If the v_y component is an increasing function of x, this tends to make the paddle wheel want to spin (positive, counter-clockwise) about the $\hat{k}$-axis. If the v_x component is a decreasing function of y, this tends to make the paddle wheel want to spin (positive, counter-clockwise) about the $\hat{k}$-axis. The net impulse to spin around the $\hat{k}$-axis is the sum of the two.
Note that this is independent of the reference frame because a constant velocity $\vec{v} =$ const. and the local acceleration $\vec{v} = \nabla f$ can subtracted because of Eq. 13-8.

Another important result is that divergence of any curl is also zero, for $\vec{v}(\vec{x}) = \vec{v}(x, y, z)$:

$$\nabla \cdot (\nabla \times \vec{v}) = 0 \tag{13-8}$$