Complex Numbers and Operations in the Complex Plane

With \( i \equiv \sqrt{-1} \), the complex numbers can be defined as the space of numbers spanned by the vectors:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ i \end{pmatrix}
\]

so that any complex number can be written as

\[
z = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ i \end{pmatrix}
\]

or just simply as

\[
z = x + iy
\]

where \( x \) and \( y \) are real numbers. \( \text{Re}z \equiv x \) and \( \text{Im}z \equiv y \).
Operations on complex numbers

Addition, subtraction, multiplication, division

Because the complex basis can be written in terms of the vectors in Equation 8-1, it is natural to plot complex numbers in two dimensions—typically these two dimensions are the “complex plane” with \((0, i)\) associated with the \(y\)-axis and \((1, 0)\) associated with the \(x\)-axis.

The reflection of a complex number across the real axis is a useful operation. The image of a reflection across the real axis has some useful qualities and is given a special name—“the complex conjugate.”

![Diagram of complex plane and complex conjugates](image.png)

Figure 8-1: Plotting the complex number \(z\) in the complex plane: The complex conjugate \((\overline{z})\) is a reflection across the real axis; the minus \((-z)\) operation is an inversion through the origin; therefore \(-\overline{z} = -z\) is equivalent to either a reflection across the imaginary axis or an inversion followed by a reflection across the real axis.

The real part of a complex number is the projection of the displacement in the real direction and also the average of the complex number and its conjugate: \(\text{Re}z = (z + \overline{z})/2\).

The imaginary part is the displacement projected onto the imaginary axis, or the complex average of the complex number and its reflection across the imaginary axis: \(\text{Im}z = (z - \overline{z})/(2i)\).
Polar Form of Complex Numbers

There are physical situations in which a transformation from Cartesian \((x, y)\) coordinates to polar (or cylindrical) coordinates \((r, \theta)\) simplifies the algebra that is used to describe the physical problem.

An equivalent coordinate transformation for complex numbers, \(z = x + iy\), has an analogous simplifying effect for multiplicative operations on complex numbers. It has been demonstrated how the complex conjugate, \(\bar{z}\), is related to a reflection—multiplication is related to a counter-clockwise rotation in the complex plane. Counter-clockwise rotation corresponds to increasing \(\theta\).

The transformations are:

\[
\begin{align*}
(x, y) &\rightarrow (r, \theta) \quad \{ \begin{array}{l}
x = r \cos \theta \\
y = r \sin \theta
\end{array} \\
(r, \theta) &\rightarrow (x, y) \quad \{ \begin{array}{l}
r = \sqrt{x^2 + y^2} \\
\theta = \arctan \frac{y}{x}
\end{array}
\end{align*}
\]

where \(\arctan \in (-\pi, \pi]\).

Multiplication, Division, and Roots in Polar Form

One advantage of the polar complex form is the simplicity of multiplication operations:
DeMoivre’s formula:

\[ z^n = r^n (\cos n\theta + i \sin n\theta) \]  
(8-5)

\[ \sqrt[n]{z} = \sqrt[n]{r} (\cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right)) \]  
(8-6)

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**Polar Form of Complex Numbers**

**Writing a function to convert to polar form**

Exponentiation and Relations to Trigonometric Functions

Exponentiation of a complex number is defined by:

\[ e^z = e^{x+iy} = e^x (\cos y + i \sin y) \]  
(8-7)

Exponentiation of a purely imaginary number advances the angle by rotation:

\[ e^{iy} = \cos y + i \sin y \]  
(8-8)

Combining Eq. 8-8 with Eq. 8-7 gives the particularly useful form:

\[ z = x + iy = re^{i\theta} \]  
(8-9)

and the useful relations (that can be obtained simply by considering the geometry of the complex plane)

\[ e^{2\pi i} = 1 \quad e^{\pi i} = -1 \quad e^{-\pi i} = -1 \quad e^{\frac{\pi}{2} i} = i \quad e^{-\frac{\pi}{2} i} = -i \]  
(8-10)
Judicious subtraction of powers in Eq. 8-8 and generalization gives the following useful relations for trigonometric functions:

\[
\begin{align*}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\
\cosh z &= \frac{e^z + e^{-z}}{2} \\
\sinh z &= \frac{e^z - e^{-z}}{2} \\
\cos iz &= \cosh z \\
\sin iz &= \sinh iz \\
\cos z &= \cosh iz \\
\sin z &= i \sinh z \\
\end{align*}
\]

(8-11)

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Numerical precision and rounding of complex numbers

Numerical and symbolic representations of complex numbers

Roots of polynomial equations

Handling complex roots of polynomial equations