Sept. 23 2005: Lecture 7:

Linear Algebra

Reading:
Kreyszig Sections: §6.5 (pp:338–41) , §6.6 (pp:341–47) , §6.7 (pp:350–57) , §6.8 (pp:358–64)

Uniqueness and Existence of Linear System Solutions

It would be useful to use the Mathematica Help Browser and look through the section in the Mathematica Book: Advanced Mathematics/ Linear Algebra/Solving Linear Equations

$$\begin{aligned}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \ldots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \ldots + A_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
A_{k1}x_1 + A_{k2}x_2 + A_{k3}x_3 + \ldots + A_{kn}x_n &= b_k \\
\vdots &= \vdots \\
A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \ldots + A_{mn}x_n &= b_m
\end{aligned} \tag{7-1}$$

$$A_{ij}x_i = b_j \tag{7-2}$$

$$\underline{A}\vec{x} = \vec{b} \tag{7-3}$$

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Properties of Determininants

$$\det \underline{A} = \det \begin{pmatrix} 1 & 2 & 1 & 1 \\ -1 & 4 & -2 & 0 \\ 1 & 2 & 4 & 5 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 14 \tag{7-4}$$

$$\underline{A}\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ gives solution } \vec{x} = \begin{pmatrix} \frac{a+b-2c+9d}{7} \\ \frac{a-d}{2} \\ \frac{13a-8b+2c-23d}{14} \\ \frac{-15a+6b+2c+19d}{14} \end{pmatrix} \tag{7-5}$$

Taking the matrix A, and replacing the third row by a linear combination ($p \times$ first row $+ q \times$ second row $+ r \times$ fourth row) of the other rows:

$$\det \underline{A_o} = \det \begin{pmatrix} 1 & 2 & 1 & 1 \\ -1 & 4 & -2 & 0 \\ p-q+r & 2p+4q & p-2q+r & p+r \\ 1 & 0 & 1 & 1 \end{pmatrix} = 0 \tag{7-6}$$

$$\underline{A_o}\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ gives no unique solution for } \vec{x} \tag{7-7}$$

No Solutions

Homogeneous Equation

$$\underline{A_o}\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ gives solutions for } \vec{x} = \begin{pmatrix} -2\chi \\ 0 \\ \chi \\ \chi \end{pmatrix} \tag{7-8}$$

Infinitely many solutions

Uniqueness of solutions to the nonhomogeneous system . . . .

$$\underline{A}\vec{x} = \vec{b} \tag{7-9}$$

Uniqueness of solutions to the homogeneous system ....

$$\underline{A}\vec{x_o} = \vec{0} \tag{7-10}$$

Adding solutions from the nonhomogeneous and homogenous systems ....

You can add any solution to the homogeneous equation (if they exist there are infinitely many of them) to any solution to the nonhomogeneous equation and the result is still a solution to the nonhomogeneous equation.

$$\underline{A}(\vec{x} + \vec{x_o}) = \vec{b} \tag{7-11}$$

Determinants

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Properties of determinants, cont'd

Determinants of random matrices

The properties of determinants ....

Vector Spaces

Consider the position vector

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{7-12}$$

The vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ can be used to generate any general position by suitable scalar multiplication and vector addition:

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7-13}$$

Thus, three dimensional real space is "spanned" by the three vectors: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. These three vectors are candidates as "basis vectors for $\Re^3$."

Consider the vectors $(a, -a, 0)$, $(a, a, 0)$, and $(0, a, a)$ for real $a \neq 0$.

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x+y}{2a} \begin{pmatrix} a \\ -a \\ 0 \end{pmatrix} + \frac{x-y}{2a} \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \frac{x-y+2z}{2a} \begin{pmatrix} 0 \\ a \\ a \end{pmatrix} \tag{7-14}$$

So $(a, -a, 0)$, $(a, a, 0)$, and $(0, a, a)$ for real $a \neq 0$ also are basis vectors and can be used to span $\Re^3$.

The idea of basis vectors and vector spaces comes up frequently in the mathematics of materials science. They can represent abstract concepts as well as shown by the following two dimensional basis set:

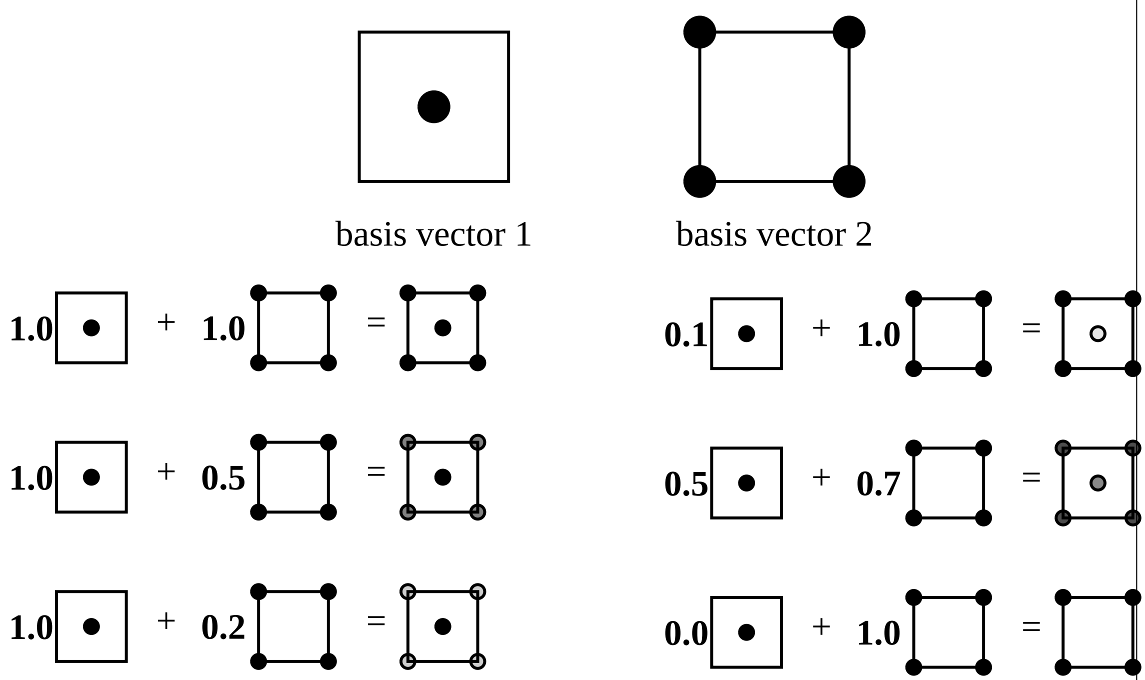

Figure 7-1: A vector space for two-dimensional CsCl structures. Any combination of center-site concentration and corner-site concentration can be represented by the sum of two basis vectors (or basis lattice). The set of all grey-grey patterns is a vector space of patterns.

Linear Transformations

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Visualization of linear transformations

1. Take a polyhedron (an octahedron, for example) and color each of the faces and display it.

2. Apply the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{7-15}$$

   to each of the vertices. Note that the transformation reflects along the z-directions across the x-y plane.

3. Apply the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{7-16}$$

   to each of the vertices.
   Using Show[]

4. Apply the matrix:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7-17}$$

   to each of the vertices. Its determinant is unity.