Massachusetts Institute of Technology

Mathematical Methods for Materials Scientists and Engineers

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Problem Set 4: Due Fri. Nov. 4, Before 5PM: email to smallen@.mit.edu

The following are this week's randomly assigned homework groups. The first member of the group is the "Homework Jefe" who will be in charge of setting up work meetings and have responsibility for turning in the group's homework notebook. If some some reason, the first member in the list is incapacitated, recalcitrant, or otherwise unavailable, then the second member should take that position. *Attention slackers:* The Jefe should include a line at the top of your notebook listing the group members that participated in the notebook's production. Group names are boldfaced text.

Aqilluqqaaq: *John Pavlish (jpavlish), EunRae Oh (eunraeoh), Jason Pelligrino (jpell19), Kelse Vandermeulen (kvander), Jill Rowehl (jillar)*

Katakartanaq: *Vladimir Tarasov (vtarasov), John Rogosic (jrogosic), Rene Chen (rrchen), Jina Kim (jinakim), Eugene Settoon (geneset)*

Masak: *Katherine Hartman (khartman), Saahil Mehra (s_mehra), Jonathon Tejada (tejada), Allison Kunz (akunz), Lisa Witmer (witmer)*

Munnguqtuq: *Richard Ramsaran (rickyr21), Maricel Delgadillo (maricela), Leanne Veldhuis (lveldhui), Kyle Yazzie (keyazzie), JinSuk Kim (jkim123), Charles Cantrell (cantrell)*

Piqsiq: *Annika Larsson (alarsson), Samuel Seong (sseong), Omar Fabian (ofabian), Michele Dufalla (mdufalla), Bryan Gortikov (bryho)*

Pukak: *Talia Gershon (tgershon), Kimberly Kam (kimkam), Katrine Sivertsen (katsiv), Lauren Oldja (oldja), Emily Gullotti (emgull)*

Individual Exercise I4-1

Kreyszig MATHEMATICA® *Computer Guide:* problem 6.14, page 78

Individual Exercise I4-2

Kreyszig MATHEMATICA® *Computer Guide:* problem 6.16, page 78

Individual Exercise I4-3

Kreyszig MATHEMATICA® *Computer Guide:* problem 7.12, page 87

Individual Exercise I4-4

Kreyszig MATHEMATICA® *Computer Guide:* problem 8.10, page 96

Individual Exercise I4-5

Kreyszig MATHEMATICA® *Computer Guide:* problem 8.22, page 96

Group Exercise G4-1

The shape of the catenary

$$y(x) = A\cosh\left(\frac{x+B}{A}\right)$$

is very important. The catenary is the shape of a flexible chain at equilibrium and the rotation of the catenary around $y = 0$ creates a surface of revolution called the *catenoid*.

In the absence of gravity, a soap film suspended between two rings with radii R_1 and R_2, axes lying along $y = 0$, and separated by distance L has a catenoid shape.

Consider a soap film suspended between two identical concentric rings of radius R and separated by distance L. Let the soap film have surface tension γ. Surface tension has units energy/area.

1. Find a parametric representation of the catenoid.
2. The mean curvuture of a surface is the sum of two curvatures. These two curvatures are obtained by slicing the surface with two orthogonal planes—creating two curves—and then using the formula for curvature for a curve. One of the curvatures is simply $1/y(x)$; the second can be obtained by using the result in Kreyszig page 443. Calculate the total mean curvature $\kappa(x)$ of the catenary and plot it.
3. Write a function that calculates the constants A and B given R and L. What are the conditions that there is one solution, two solutions, no solutions?
4. Write a function that calculates the total surface energy, $E(R, L)$, of a soap film.

   The equation for the area of a surface of revolution is:

   $$A[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + (\frac{dy}{dx})^2} dx$$

   Plot the normalized energy surface(s) $E(R, L)/(\gamma RL)$.

Group Exercise G4-2

The diffusion equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c$$

describes how the concentration field $c(\vec{r},t)$ changes with time proportional to spatial second derivatives. A solution to the diffusion equation requires that *initial conditions* and *boundary conditions* be specified. Boundary conditions specify how $c(\vec{r},t)$ behaves at particular points in space for all times. Initial conditions specify how $c(\vec{r},t)$ behaves throughout all space at a particular time.

For some boundary conditions (BCs) and initial conditions (ICs), it is possible to write a solution to the diffusion equation in terms of an integral. For solutions in the infinite domain, the following BCs and ICs are a pair of such conditions,

$$c(x=\pm\infty, y=\pm\infty, z=\pm\infty, t) = 0 \tag{1}$$

$$c(x,y,z,t=0) = \begin{cases} c_0 & \text{if}|x| \le \frac{a}{2} \text{ and } |y| \le \frac{b}{2} \text{ and } |z| \le \frac{c}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where a, b, and c are finite (i.e., the initial conditions have uniform concentration, c_0, inside a rectangular box and zero outside.

1. Show that
$$c(x,y,z,t) = \int_{\frac{-a}{2}}^{\frac{a}{2}} \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-c}{2}}^{\frac{c}{2}} \frac{c_0 d\zeta d\eta d\chi}{(4\pi Dt)^{3/2}} e^{-\frac{(x-\chi)^2+(y-\eta)^2+(z-\zeta)^2}{4Dt}} \tag{3}$$
always satisfies the diffusion equation (independent of BCs and ICs).
2. Show that Eq. 3 always satisfies the boundary conditions, independent of the ICs.
3. Find the closed form of $c(x,y,z,t)$ that satisfies both Eq. 1 and 2.
4. Show by a graphical means that $c(x,y,z,t)$ plausibly approaches the ICs (Eq. 2) as $t \to 0$.
5. Show that the total number of atoms is conserved for $c(x,y,z,t)$.

Group Exercise G4-3

The potential energy of two small magnetic dipoles $\vec{\mu_1}$ and $\vec{\mu_2}$ located at points $\vec{r_1}$ and $\vec{r_2}$ are given by

$$U(\vec{r_1},\vec{r_2}) = \frac{\mu_o}{4\pi}\left\{\frac{\vec{\mu_1}\cdot\vec{\mu_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{3[\vec{\mu_1}\cdot(\vec{r_1}-\vec{r_2})][\vec{\mu_2}\cdot(\vec{r_1}-\vec{r_2})]}{|\vec{r_1}-\vec{r_2}|^5}\right\}$$

Suppose the first magnetic dipole is located at the origin and points towards the z-direction.

1. Illustrate the potential energy of the two-dipole system as a function of the second magnet's position $\vec{r_2}$ if it is also directed towards the z-direction.

2. Illustrate the potential energy of the two-dipole system if the second magnet is fixed at the location $\vec{r_2}$ but is rotated by θ about the normal to the plane containing both magnets and the z-axis.

3. Illustrate the potential energy of the two-dipole system as a function *both* the second magnet's position $\vec{r_2}$ and its rotation θ about the normal to the plane containing both magnets and the z-axis.

4. Suppose the second magnet is moved along a trajectory, $(x, y, z) = r_0(\cos(2\pi t), \sin(2\pi t), 0)$, and the magnet is always directed towards the trajectory's tangent. Calculate and illustrate the potential energy and the rate of work done on the system as a function of time.

5. **Extra Credit:** Suppose the two magnets are immersed in a viscous fluid and the first magnet is fixed as above. The rate of rotation is given by (approximately)

$$\frac{d\theta}{dt} = \frac{\tau}{4\pi\eta R^2 L}$$

   where R and L are the radius and length of the cylindrical magnet and η is the viscosity in the fluid medium. τ is the torque applied to the magnet.

   The velocity is given by (very approximately)

$$\frac{d\vec{r}}{dt} = \frac{\vec{F}}{6\pi\eta R}$$

   where $\vec{F}$ is the force applied to the magnet.

   Graphically illustrate the position of the rod as a function of time, if the rod is initially at rest at $t = 0$ and located at $\vec{r} = r_0$ for the following initial inclination angles:

$$\theta = (0°, 1°, 45°, 89°, 90°, 91°, 135°, 179°, 180°, 181°, 225°, 269°, 270°, 271°, 315°, 359°)$$