Solution to Sample Problem in Recitation 4

In this recitation, we will look at:

- Review:
  - Types of Work: Electric and Magnetic
  - Brief Review of Tensors
  - Elastic Tensors
  - Elastic Strain Energy
  - Note: Happy Meal Deal.....
- Questions regarding homework
- Sample Problems

Problem 1

This problem was given as a Problem Set last year.

The bulk modulus, K, of an isotropic linear elastic solid is defined by the dilation, $\frac{\triangle V}{V_0}$, response to hydrostatic pressure, P:

$$\frac{\triangle V}{V_0} = \frac{V - V_0}{V_0} = -\frac{P}{K} \tag{1}$$

- Typical values of K for an ionic crystal are about 100 GPa.
- The electric permittivity of vacuum, κ_0, is $8.85 \times 10^{-12}\ \frac{C^2}{Jm}$. Typical values of the dielectric susceptibility,χ,($\overrightarrow{P} = \kappa_0\ \chi\ \overrightarrow{E}$), of an ionic crystal are about 50 (unitless).
- The magnetic permittivity of vacuum, μ_0, is $4\ \pi\ \times 10^{-7}\ \frac{T^2}{Jm^3}$. The magnetic susceptibility, ψ ($\overrightarrow{I} = \mu\ \psi\ \overrightarrow{H}$), of a typical paramagnetic ionic crystal is about 10 (unitless).

Calculate all the ratios of: stored elastic energy, stored polarization energy and stored magnetic energy in a typical ionic crystal at 1 atm, 220 $\frac{V}{m}$ and in earth's magnetic field.

Solution:
V_0 everywhere is given in m^3.

Elastic work done on the system:

- We know that $W_{el} = -\int_{V_i}^{V_f} P\ dV$. Additionally, $\frac{\triangle V}{V_0} = -\frac{P}{\kappa}$ or $\frac{\partial V}{\partial P} = -\frac{V_0}{K}$. Using $dV = \frac{\partial V}{\partial P}\ dP$ we get:

$$W_{el} = -\int_{V_i}^{V_f} P\ dV = \ -\int_{P_i}^{P_f} P\frac{\partial V}{\partial P}dP = -\int_{P_i}^{P_f} -P\ \frac{P\ V_0}{K}\ dP = \frac{P^2\ V_0}{2\ K} \tag{2}$$

Note that this IS NOT a constant pressure process. The total elastic work is:

$$W_{el} = \frac{P^2\ V_0}{2\ K} = \frac{(10^5 Pa)^2\ V_0}{2\ 10^{11} Pa} = 0.05 V_0\ J \tag{3}$$

Electric work done on the system:

As this is an isotropic solid all the vector and tensor quantities can be treated as scalars. The polarization work done on the system is $W_{pol} = \int_{D_i}^{D_f} V_0 E \cdot dD$. Where $D = \kappa_0(1+\chi)E$ and $dD = \kappa_0(1+\chi)dE$.

$$W_{pol} = \int_0^E\ V_0\ \kappa_0(1+\chi)\ E\ dE = \frac{V_0\ \kappa_0(1+\chi)\ E^2}{2} \tag{4}$$

$$W_{pol} = \frac{V_0\ 8.85\times 10^{-12}\frac{C^2}{Jm}\ (1+50)\ (220\frac{V}{m})^2}{2} = 1.09\times 10^{-5}\ V_0 J \tag{5}$$

Magnetic work done on the system:

We know that the magnetic field on earth's surface is $10^{-4}T$. The magnetic work done on the system is $W_{mag} = \int_{B_i}^{B_f} V_0\ \ H\cdot dB$. Where $B = \mu_0\ (1+\psi)\ H$ and $dB = \mu_0\ (1+\psi)dH$.

$$W_{mag} = \int_0^H V_0\ \mu_0\ (1+\psi)H\ dH = \frac{V_0\ \mu_0(1+\psi)\ H^2}{2} \tag{6}$$

$$W_{mag} = \frac{V_0\ 4\pi\times 10^{-7}\frac{J}{T^2 m^3}\ (1+10)\ (10^{-4}T)^2}{2} = 6.91\times 10^{-14} V_0 J \tag{7}$$

Therefore:

$$\frac{W_{el}}{W_{pol}} = 4587 \tag{8}$$

and

$$\frac{W_{el}}{W_{mag}} = 7.24 \times 10^{11} \tag{9}$$