Solution to Sample Problem in Recitation 4

In this recitation, we will look at:

- Review:
  - Types of Work: Electric and Magnetic
  - Brief Review of Tensors
  - Elastic Tensors
  - Elastic Strain Energy
  - Note: Happy Meal Deal....

- Questions regarding homework

- Sample Problems

**Problem 1**

This problem was given as a Problem Set last year.

The bulk modulus, $K$, of an isotropic linear elastic solid is defined by the dilation, $\frac{\Delta V}{V_0}$, response to hydrostatic pressure, $P$:

$$\frac{\Delta V}{V_0} = \frac{V - V_0}{V_0} = -\frac{P}{K} \quad (1)$$

- Typical values of $K$ for an ionic crystal are about 100 GPa.

- The electric permittivity of vacuum, $\kappa_0$, is $8.85 \times 10^{-12} \ \frac{F}{m}$. Typical values of the dielectric susceptibility, $\chi_e(\vec{B} = \kappa_0 \ \vec{E})$, of an ionic crystal are about 50 (unitless).

- The magnetic permittivity of vacuum, $\mu_0$, is $4 \pi \times 10^{-7} \ \frac{T^2}{Jm^2}$. The magnetic susceptibility, $\psi \ (\vec{T} = \mu_\psi \vec{H})$, of a typical paramagnetic ionic crystal is about 10 (unitless).

Calculate all the ratios of: stored elastic energy, stored polarization energy and stored magnetic energy in a typical ionic crystal at 1 atm, 220 $\frac{V}{m}$ and in earth’s magnetic field.
Solution:

\( V_0 \) everywhere is given in \( m^3 \).

Elastic work done on the system:

- We know that \( W_{el} = - \int_{V_i}^{V_f} P \, dV \). Additionally, \( \frac{\Delta V}{V_0} = -\frac{P}{K} \) or \( \frac{\partial V}{\partial P} = -\frac{V_0}{\kappa} \). Using \( dV = \frac{\partial V}{\partial P} \, dP \) we get:

\[
W_{el} = - \int_{V_i}^{V_f} P \, dV = - \int_{P_i}^{P_f} P \frac{\partial V}{\partial P} \, dP = - \int_{P_i}^{P_f} P \frac{V_0}{K} \, dP = \frac{P^2}{2} \frac{V_0}{K}
\]

(2)

Note that this IS NOT a constant pressure process. The total elastic work is:

\[
W_{el} = \frac{P^2}{2} \frac{V_0}{K} = \frac{(10^5 Pa)^2 V_0}{2 \times 10^{11} Pa} = 0.05 V_0 \, J
\]

(3)

Electric work done on the system:

As this is an isotropic solid all the vector and tensor quantities can be treated as scalars. The polarization work done on the system is \( W_{pol} = \int_{D_i}^{D_f} V_0 E \cdot dD \). Where \( D = \kappa_0 (1 + \chi) E \) and \( dD = \kappa_0 (1 + \chi) dE \).

\[
W_{pol} = \int_{E}^{E} V_0 \kappa_0 (1 + \chi) E \, dE = \frac{V_0 \kappa_0 (1 + \chi) E^2}{2}
\]

(4)

\[
W_{pol} = V_0 \times 10^{-12} \frac{C^2}{m} (1 + 50) (220 \frac{V}{m})^2 = 1.09 \times 10^{-5} V_0 \, J
\]

(5)

Magnetic work done on the system:

We know that the magnetic field on earth’s surface is \( 10^{-4} T \). The magnetic work done on the system is \( W_{mag} = \int_{B_i}^{B_f} V_0 H \cdot dB \). Where \( B = \mu_0 (1 + \psi) H \) and \( dB = \mu_0 (1 + \psi) dH \).

\[
W_{mag} = \int_{H}^{H} V_0 \mu_0 (1 + \psi) H \, dH = \frac{V_0 \mu_0 (1 + \psi) H^2}{2}
\]

(6)

\[
W_{mag} = \frac{V_0 4 \pi \times 10^{-7} \frac{J}{T^2 m^3} (1 + 10) (10^{-4} T)^2}{2} = 6.91 \times 10^{-14} V_0 J
\]

(7)
Therefore:

\[
\frac{W_{el}}{W_{pol}} = 4587 \quad (8)
\]

and

\[
\frac{W_{el}}{W_{mag}} = 7.24 \times 10^{11} \quad (9)
\]